An example in multi-parametric geometry of numbers

Damien Roy

University of Ottawa

Diophantine approximation and related fields

York University

June 9–13, 2025

Dedicated to Bertrand Russell
and all mathematicians working for peace

https:\\mysite.science.uottawa.ca\droy\talks\York_2025_Beamer.pdf

General framework (dimension 3)

Let $A = (a_{i,j}) \in \mathrm{GL}_3(\mathbb{R})$ and let $\mathbf{q} = (q_1, q_2, q_3) \in \mathbb{R}^3$. Consider the parallelepiped

$$\mathcal{C}_A(\mathbf{q}) : \begin{cases} |a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3| \leq e^{-q_1} \\ |a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3| \leq e^{-q_2} \\ |a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3| \leq e^{-q_3} \end{cases}$$

For each $i = 1, 2, 3$, the logarithm of its i-th minimum with respect to $\mathbb{Z}^3$ is the smallest $t \in \mathbb{R}$, denoted $L_{A,i}(\mathbf{q})$ such that

$$e^t\mathcal{C}_A(\mathbf{q}) : \begin{cases} |a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3| \leq e^{t-q_1} \\ |a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3| \leq e^{t-q_2} \\ |a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3| \leq e^{t-q_3} \end{cases}$$

contains at least i linearly independent points $\mathbf{x}$ in $\mathbb{Z}^3$. We form the map

$$\begin{array}{rccc} \mathbf{L}_A : & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ & \mathbf{q} & \longmapsto & (L_{A,1}(\mathbf{q}), L_{A,2}(\mathbf{q}), L_{A,3}(\mathbf{q})) \end{array}$$

? Determine $\mathbf{L}_A$ up to bounded error on $\mathbb{R}^3$.

? Characterize the set of all maps $\mathbf{L}_A$ modulo bounded functions on $\mathbb{R}^3$.

Since $\log \mathrm{vol}(\mathcal{C}_A(\mathbf{q})) = -(q_1 + q_2 + q_3) + \mathcal{O}_A(1)$, Minkowski's convex body theorem gives

$$L_{A,1}(\mathbf{q}) + L_{A,2}(\mathbf{q}) + L_{A,3}(\mathbf{q}) = q_1 + q_2 + q_3 + \mathcal{O}_A(1).$$

Since $L_{A,1}(\mathbf{q}) \leq L_{A,2}(\mathbf{q}) \leq L_{A,3}(\mathbf{q})$, we deduce that

$$L_{A,1}(\mathbf{q}) \leq (q_1 + q_2 + q_3)/3 + \mathcal{O}_A(1).$$

Example

Let K be a totally real cubic number field and let $A = (\sigma_i(\omega_j))$ where

- $(\omega_1, \omega_2, \omega_3)$ is a basis of the ring of integers of K,
- $(\sigma_1, \sigma_2, \sigma_3)\colon K \to \mathbb{R}^3$ is the canonical embedding of K.

Then

$$\sup_{\mathbf{q} \in \mathbb{R}^3} \max_{1 \leq i \leq 3} |L_{A,i}(\mathbf{q}) - (q_1 + q_2 + q_3)/3| < \infty.$$

Proof

Let $t = L_{A,1}(\mathbf{q})$ and let $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{Z}^3$ be a non-zero integer point of

$$e^t \mathcal{C}_A(\mathbf{q}) \; : \; \begin{cases} |\sigma_1(\omega_1)x_1 + \sigma_1(\omega_2)x_2 + \sigma_1(\omega_3)x_3| \leq e^{t-q_1} \\ |\sigma_2(\omega_1)x_1 + \sigma_2(\omega_2)x_2 + \sigma_2(\omega_3)x_3| \leq e^{t-q_2} \\ |\sigma_3(\omega_1)x_1 + \sigma_3(\omega_2)x_2 + \sigma_3(\omega_3)x_3| \leq e^{t-q_3} \end{cases}$$

Set $\alpha = x_1\omega_1 + x_2\omega_2 + x_3\omega_3$. By hypothesis

$$|\sigma_i(\alpha)| \leq e^{t-q_i} \quad \text{for each } i = 1, 2, 3,$$

thus

$$1 \leq |\mathrm{Norm}_{K/\mathbb{Q}}(\alpha)| = \prod_{i=1}^{3} |\sigma_i(\alpha)| \leq e^{3t-q_1-q_2-q_3},$$

and so

$$q_1 + q_2 + q_3 \leq 3t = 3L_{A,1}(\mathbf{q}) \leq \sum_{i=1}^{3} L_{A,i}(\mathbf{q}) = q_1 + q_2 + q_3 + \mathcal{O}_A(1).$$

Specific framework

- We restrict to $A = \begin{pmatrix} 1 & 0 & 0 \\ \xi_1 & -1 & 0 \\ \xi_2 & 0 & -1 \end{pmatrix}$ where $\boldsymbol{\xi} = (1, \xi_1, \xi_2) \in \mathbb{R}^3$.
- Since $e^t \mathcal{C}_{\boldsymbol{\xi}}(q_1, q_2, q_3) = \mathcal{C}_{\boldsymbol{\xi}}(q_1 - t, q_2 - t, q_3 - t)$, we may fix $q_1 = 0$.
  (The standard normalization is $q_1 + q_2 + q_3 = 0$.)

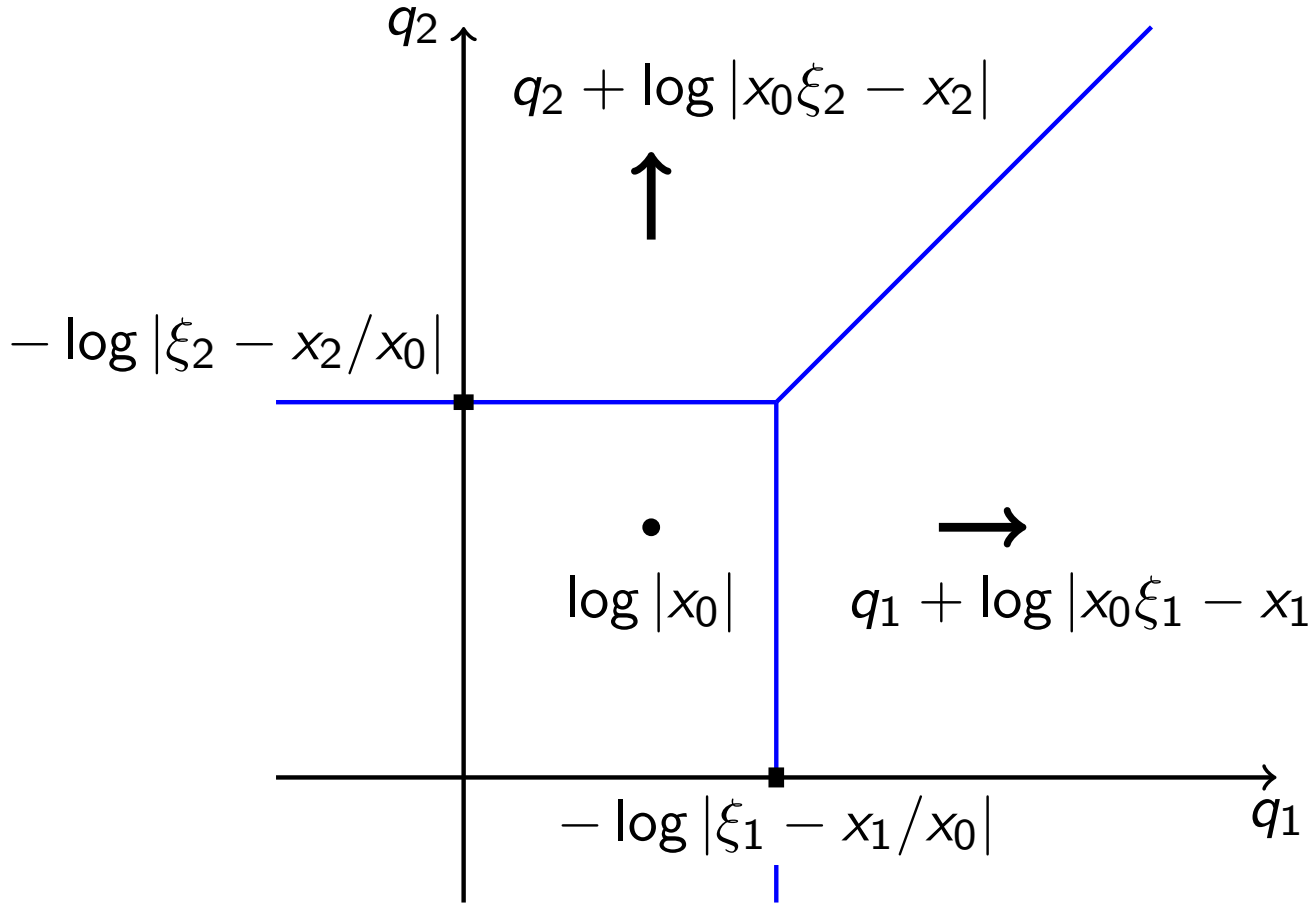
- Thus, we work with $\mathcal{C}_{\boldsymbol{\xi}}(q_1, q_2)$: $\begin{cases} |x_0| \leq 1 \\ |x_0\xi_1 - x_1| \leq e^{-q_1} \\ |x_0\xi_2 - x_2| \leq e^{-q_2} \end{cases}$
- $L_{\boldsymbol{\xi},i}(q_1, q_2)$ = smallest $t \in \mathbb{R}$ for which the conditions

$$|x_0| \leq e^t, \quad |x_0\xi_1 - x_1| \leq e^{t-q_1}, \quad |x_0\xi_2 - x_2| \leq e^{t-q_2}$$

admit at least i linearly independent solutions $\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{Z}^3$.

The trajectory of a non-zero point $\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{Z}^3$ is the map $L_{\mathbf{x}} : \mathbb{R}^2 \to \mathbb{R}$ given by

$$L_{\mathbf{x}}(q_1, q_2) = \text{the smallest } t \text{ such that } \mathbf{x} \in e^t \mathcal{C}_{\boldsymbol{\xi}}(q_1, q_2)$$
$$= \max\{\log|x_0|, q_1 + \log|x_0\xi_1 - x_1|, q_2 + \log|x_0\xi_2 - x_2|\}.$$



(if $x_0 \neq 0$)

Basic tool

We denote by

$$\Phi\colon \mathbb{R}^3 \to \{(t_0, t_1, t_2) \in \mathbb{R}^3 ;\ t_0 \le t_1 \le t_2\}$$

the map that puts the coordinates of a point in non-decreasing order.

Let $\mathbf{q} = (q_1, q_2) \in \mathbb{R}^2$ and let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{Z}^3$ be **linearly independent**. Then

$$\left\|\Phi(L_{\mathbf{x}_1}(\mathbf{q}), L_{\mathbf{x}_2}(\mathbf{q}), L_{\mathbf{x}_3}(\mathbf{q})) - \mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q})\right\|_\infty \le \sum_{i=1}^{3} L_{\mathbf{x}_i}(\mathbf{q}) - (q_1 + q_2) + \mathcal{O}_{\boldsymbol{\xi}}(1).$$

Proof.

$\mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q}) \le \Phi(L_{\mathbf{x}_1}(\mathbf{q}), L_{\mathbf{x}_2}(\mathbf{q}), L_{\mathbf{x}_3}(\mathbf{q}))$ coordinate-wise. Thus,

$$\left\|\Phi(L_{\mathbf{x}_1}(\mathbf{q}), L_{\mathbf{x}_2}(\mathbf{q}), L_{\mathbf{x}_3}(\mathbf{q})) - \mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q})\right\|_\infty \le \sum_{i=1}^{3} L_{\mathbf{x}_i}(\mathbf{q}) - \sum_{i=1}^{3} L_{\boldsymbol{\xi},i}(\mathbf{q}).$$

□

Example

Let $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$, $\mathbf{e}_3 = (0,0,1)$ denote the elements of the canonical basis of $\mathbb{Z}^3$. We find

$$\begin{aligned} L_{\mathbf{e}_1}(\mathbf{q}) &= \max\{0, q_1 + \log|\xi_1|, q_2 + \log|\xi_2|\}, \\ L_{\mathbf{e}_2}(\mathbf{q}) &= \max\{-\infty, q_1, q_2 - \infty\} = q_1, \\ L_{\mathbf{e}_3}(\mathbf{q}) &= \max\{-\infty, q_1 - \infty, q_2\} = q_2. \end{aligned}$$

If $q_1 \leq 0$ and $q_2 \leq 0$, then $L_{\mathbf{e}_1}(\mathbf{q}) = \mathcal{O}_{\boldsymbol{\xi}}(1)$, so

$$\Phi(L_{\mathbf{e}_1}(\mathbf{q}), L_{\mathbf{e}_2}(\mathbf{q}), L_{\mathbf{e}_3}(\mathbf{q})) = \Phi(0, q_1, q_2) + \mathcal{O}_{\boldsymbol{\xi}}(1),$$

$$\sum_{i=1}^{3} L_{\mathbf{e}_i}(\mathbf{q}) = q_1 + q_2 + \mathcal{O}_{\boldsymbol{\xi}}(1).$$

Thus:

If $q_1 \leq 0$ and $q_2 \leq 0$, then $\mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q}) = \Phi(q_1, q_2, 0) + \mathcal{O}_{\boldsymbol{\xi}}(1)$.

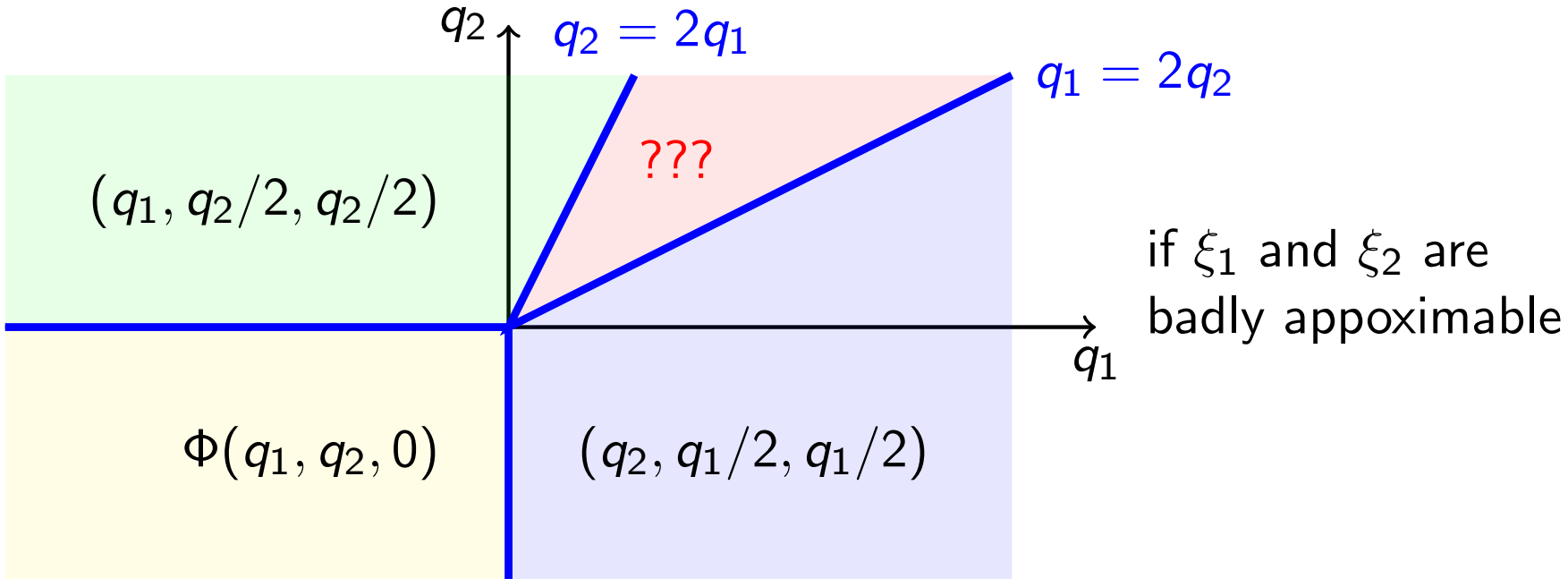
Suppose that ξ_1 is badly approximable

$$\text{i.e.} \quad \nu(\xi_1) := \liminf_{n\to\infty} n\|n\xi_1\| > 0,$$

then, for $\mathbf{q} = (q_1, q_2)$ with $q_1 \geq 0$ and $q_2 \leq q_1/2$, we have

$$\mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q}) = (q_2, q_1/2, q_1/2) + \mathcal{O}_{\boldsymbol{\xi}}(1).$$

Proof. Take $\mathbf{x}_1 = (n, m, [n\xi_2])$, $\mathbf{x}_2 = (n', m', [n'\xi_2])$ and $\mathbf{x}_3 = (0, 0, 1)$ where m/n and m'/n' are consecutive convergents of ξ_1 with $\log n = q_1/2 + \mathcal{O}_{\xi_1}(1)$.



Littlewood's conjecture

For any $\boldsymbol{\xi} = (1, \xi_1, \xi_2) \in \mathbb{R}^3$ and any $b > 0$, there exists $\mathbf{q} = (q_1, q_2) \in \mathbb{R}^2$ with $0 \leq q_1/2 \leq q_2 \leq 2q_1$ such that

$$L_{\boldsymbol{\xi},1}(\mathbf{q}) \leq (q_1 + q_2)/3 - b.$$

Goal of this talk

Compute $\mathbf{L}_{\boldsymbol{\xi}}(q_1, q_2)$ up to bounded error in the region where $q_1 \geq 0$ and $q_2 \leq q_1$ for $\boldsymbol{\xi} = (1, \xi, \xi^2)$ where ξ is an extremal real number.

The plan is to describe

I the particular extremal numbers ξ,

II the approximation function (2-parameter 3-system),

III the triple of points which approximate the three minima.

Part I. The numbers.

I.1. Notation.

- The norm $\|A\|$ of a matrix A with real coefficients is the maximum of the absolute values of its entries.
- A matrix $A \in \mathrm{Mat}_{2\times 2}(\mathbb{Z})$ is said to be primitive if $A \neq 0$ and if the gcd of its coefficients is 1.
- We identify any $\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{Z}^3$ with the symmetric matrix

$$\mathbf{x} = \begin{pmatrix} x_0 & x_1 \\ x_1 & x_2 \end{pmatrix} \in \mathrm{Mat}_{2\times 2}(\mathbb{Z}).$$

Then, for $\xi \in \mathbb{R}$, we have

$$\|(\xi, -1)\mathbf{x}\| = \max\{|x_0\xi - x_1|,\ |x_1\xi - x_2|\}.$$

Thus,

$$\begin{aligned} |x_0\xi^2 - x_2| &= |(x_0\xi - x_1)\xi + (x_1\xi - x_2)| \\ &= |x_1\xi - x_2| + \mathcal{O}(\|(\xi, -1)\mathbf{x}\|). \end{aligned}$$

I.2. Fibonacci sequences

Definition. A Fibonacci sequence in a monoid $\mathcal{M}$ is a sequence $(w_i)_{i\geq 1}$ in $\mathcal{M}$ such that $w_{i+2} = w_{i+1}w_i$ for each $i \geq 1$.

Example 1. The set $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ is a monoid under addition.
We denote by $(F_n)_{n\geq 1} = (1, 2, 3, 5, 8, \dots)$ the usual Fibonacci sequence in $\mathbb{N}$ with $F_1 = 1$, $F_2 = 2$ and $F_{i+2} = F_{i+1} + F_i$ for each $i \geq 1$.

Example 2. Let E^* = monoid of words on an alphabet E.
The Fibonacci sequence in $\{a, b\}^*$ starting with $w_1 = a$ and $w_2 = ab$ has

$$w_3 = aba,\ \ w_4 = abaab,\ \ w_5 = abaababa,\ \ \dots$$

It converges to the Fibonacci word on a, b,

$$f_{a,b} := w_\infty := \lim_{i\to\infty} w_i = abaababaabaab\dots$$

The map $|\ |\colon \{a, b\}^* \to \mathbb{N}$ sending a word w to its length $|w|$ is a morphism of monoids. We have $|w_i| = F_i$ for each $i \geq 1$.

I.3. Extremal numbers

Set $\boxed{\gamma = (1+\sqrt{5})/2 \simeq 1.618}$ $\Rightarrow$ $1/\gamma = \gamma - 1 \simeq 0.618$

Theorem (Davenport and Schmidt, 1969)

Let $\xi \in \mathbb{R}$ such that $1, \xi, \xi^2$ are linearly independent over $\mathbb{Q}$. There exists $c_1 = c_1(\xi) > 0$ such that, **for arbitrarily large values of X**, *the conditions*

$$|x_0| \leq X, \quad |x_0\xi - x_1| \leq c_1 X^{-1/\gamma}, \quad |x_0\xi^2 - x_2| \leq c_1 X^{-1/\gamma}$$

have **no non-zero solution** $\mathbf{x} = (x_0, x_1, x_2)$ *in $\mathbb{Z}^3$.*

Corollary (Davenport and Schmidt, 1969)

For $\xi \in \mathbb{R}$ as above, there exists $c_2 = c_2(\xi) > 0$ such that

$$|\xi - \alpha| \leq c_2 H(\alpha)^{-\gamma-1}$$

for infinitely many algebraic integers α of degree at most 3 over $\mathbb{Q}$.

Here, $H(\alpha) =$ maximum of the absolute values of the coefficients of the irreducible polynomial of α over $\mathbb{Z}$.

Definition. We say that a real number ξ is **extremal** if $1, \xi, \xi^2$ are linearly independent over $\mathbb{Q}$ and if there exists a constant $c > 0$ such that, **for each large enough** X, the conditions

$$|x_0| \leq X, \quad |x_0\xi - x_1| \leq cX^{-1/\gamma}, \quad |x_0\xi^2 - x_2| \leq cX^{-1/\gamma}$$

admit **a non-zero solution** $\mathbf{x} = (x_0, x_1, x_2)$ in $\mathbb{Z}^3$.

Theorem (R. 2003)

The set of extremal real numbers is countably infinite.

- These numbers are transcendental over $\mathbb{Q}$.
- If ξ is extremal and $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q})$, then $A.\xi := \dfrac{a\xi + b}{c\xi + d}$ is extremal.
- **Open problem:** Are all extremal real numbers badly approximable?

I.4. Characterization of extremal numbers

Theorem (R. 2004)

1. *For each extremal $\xi \in \mathbb{R}$, there exist an unbounded sequence of primitive symmetric matrices $\mathbf{x}_i = \begin{pmatrix} x_{i,0} & x_{i,1} \\ x_{i,1} & x_{i,2} \end{pmatrix} \in Mat_{2\times 2}(\mathbb{Z})$ and a matrix $M \in Mat_{2\times 2}(\mathbb{Z})$ with ${}^tM \neq \pm M$ such that, for each $i \geq 1$,*

(E1) $\|(\xi, -1)\mathbf{x}_i\| = \max\{|x_{i,0}\xi - x_{i,1}|, |x_{i,1}\xi - x_{i,2}|\} \asymp \|\mathbf{x}_i\|^{-1}$,

(E2) $\|\mathbf{x}_{i+1}\| \asymp \|\mathbf{x}_i\|^{\gamma}$,

(E3) $1 \leq |\det \mathbf{x}_i| \ll 1$,

(E4) $\mathbf{x}_{i+2} \propto \mathbf{x}_{i+1} M_{i+1} \mathbf{x}_i$ *where* $M_i = \begin{cases} M & \text{if } i \text{ is even,} \\ {}^tM & \text{if } i \text{ is odd.} \end{cases}$

The sequence $(\mathbf{x}_i)_{i\geq 1}$ is uniquely determined by ξ up to its first terms and up to multiplication of its terms by ± 1.

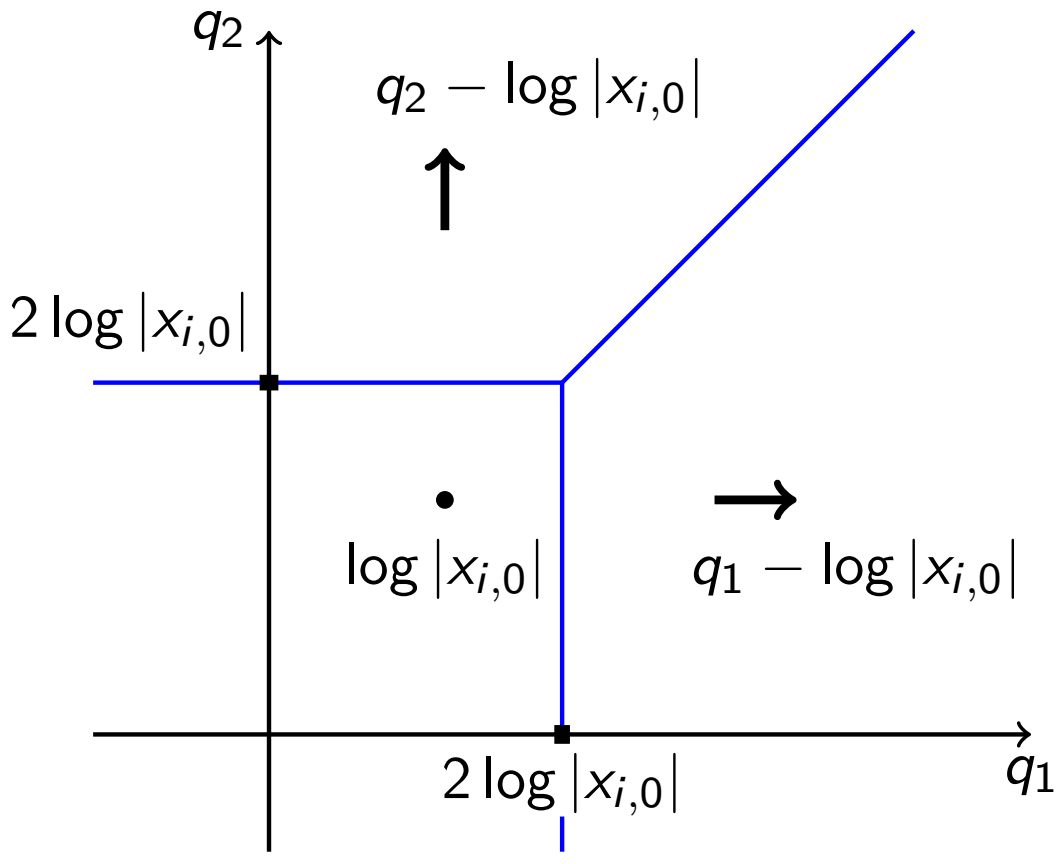
2. *Conversely, for any unbounded sequence $(\mathbf{x}_i)_{i\geq 1}$ of primitive symmetric matrices $(\mathbf{x}_i)_{i\geq 1}$ in $Mat_{2\times 2}(\mathbb{Z})$ which satisfies (E2)–(E4), there is an extremal $\xi \in \mathbb{R}$ for which (E1) holds as well.*

I.5. Trajectory of $\mathbf{x}_i$

It can be shown that, for each $i \geq 1$,

$$|x_{i,0}\xi - x_{i,1}| \asymp |x_{i,0}\xi^2 - x_{i,2}| \asymp |x_{i,0}|^{-1},$$

$$\Rightarrow \boxed{L_{\mathbf{x}_i}(q_1, q_2) = \max\{\log|x_{i,0}|, q_1 - \log|x_{i,0}|, q_2 - \log|x_{i,0}|\} + \mathcal{O}_\xi(1).}$$



I.6. Extremal numbers of $\mathrm{SL}_2(\mathbb{Z})$-type [R. 2008]

- They are the extremal real numbers ξ associated to a sequence of symmetric matrices $(\mathbf{x}_i)_{i\geq 1}$ in $\mathrm{SL}_2(\mathbb{Z})$.
- Then, we also have $M \in \mathrm{SL}_2(\mathbb{Z})$.
- The conditions on $(\mathbf{x}_i)_{i\geq 1}$ are reduced to

$$\mathbf{x}_{i+2} = \mathbf{x}_{i+1} M_{i+1} \mathbf{x}_i = \mathbf{x}_i M_i \mathbf{x}_{i+1} \text{ where } M_i = \begin{cases} M & \text{if } i \text{ is even,} \\ {}^tM & \text{if } i \text{ is odd,} \end{cases}$$

$$\|\mathbf{x}_{i+2}\| \asymp \|\mathbf{x}_{i+1}\| \, \|\mathbf{x}_i\| \to \infty.$$

- Then we have $\|\mathbf{x}_{i+1}\| \asymp \|\mathbf{x}_i\|^\gamma$ and ξ is characterized by

$$\|(\xi, -1)\mathbf{x}_i\| \asymp \|\mathbf{x}_i\|^{-1} \asymp \|\mathbf{x}_{i+1}\|^{-1/\gamma},$$

- The number ξ is badly approximable.
- The matrices $W_i = \mathbf{x}_i M_i$ form a Fibonacci sequence in $\mathrm{SL}_2(\mathbb{Z})$, i.e.

$$W_{i+2} = W_{i+1} W_i \quad \text{for each } i \geq 1.$$

I.7. Choice of ξ

Objective

We want to estimate $\mathbf{L}_{\boldsymbol{\xi}}$ for points $\boldsymbol{\xi} = (1, \xi, \xi^2)$ where ξ is an extremal real number of $\mathrm{SL}_2(\mathbb{Z})$-type with associated matrix

$$M = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}.$$

Knowned properties of such ξ

- There exists $c = c(\xi) > 0$ such that $|\xi - \alpha| \geq cH(\alpha)^{-\gamma^2}$ for each algebraic integer α of degree ≤ 3 [R. 2003].
- Similar results for approximation by algebraic integers of degree ≤ 4 and trace 0, etc [R.-Zelo 2011].
- $\liminf_{n\to\infty} n\|n\xi\| = 1/3$ is largest possible for a non-quadratic irrational number [R. 2011].

A specific example

There is a sequence of symmetric matrices $(\mathbf{x}_i)_{i\geq 1}$ in $\mathrm{SL}_2(\mathbb{Z})$ starting with

$$\mathbf{x}_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},\ \mathbf{x}_2 = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix},\ \mathbf{x}_3 = \begin{pmatrix} 29 & 17 \\ 17 & 10 \end{pmatrix},\ \mathbf{x}_4 = \begin{pmatrix} 433 & 254 \\ 254 & 149 \end{pmatrix},\ \cdots$$

and obeying recurrence relation

$$\mathbf{x}_{i+2} = \mathbf{x}_{i+1} M_{i+1} \mathbf{x}_i = \mathbf{x}_i M_i \mathbf{x}_{i+1} \quad (i \geq 1)$$

where

$$M_i = \begin{pmatrix} 3 & (-1)^i \\ (-1)^{i+1} & 0 \end{pmatrix} \quad (i \geq 1).$$

It satisfies

$$\|(\xi, -1)\mathbf{x}_i\| \asymp \|\mathbf{x}_i\|^{-1} \asymp \|\mathbf{x}_{i+1}\|^{-1/\gamma}$$

for the real number ξ whose continued fraction expansion is

$$\xi = [0, 1, 1, 2, 2, 1, 1, 2, 2, 2, 2, 1, 1, 2, 2, \ldots] = [0, \mathbf{1}, f_{\mathbf{2},\mathbf{1}}]$$

where $f_{\mathbf{2},\mathbf{1}}$ is the Fibonacci word on $\mathbf{2} = (2, 2)$ and $\mathbf{1} = (1, 1)$.

Part II. The function.

II.1. The maps ι and α.

Definition. Set

$$\mathcal{F} = \{0, F_1, F_2, F_3, \dots\} = \{0, 1, 2, 3, 5, 8, 13, 21, \dots\}.$$

We define a map $\iota\colon \mathbb{N} \to \mathbb{N}$ by

$$\iota(x) = \begin{cases} x & \text{if } x \in \mathcal{F}, \\ x - 2F_{k-2} & \text{if } F_k < x < F_{k+1} \text{ for some } k \geq 3. \end{cases}$$

We also define $\alpha\colon \mathbb{N} \to \mathcal{F}$ by

$$\alpha(x) = \lim_{k\to\infty} \iota^k(x)$$

where ι^k denotes the k-th iterate of ι.

x	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$\iota(x)$	0	1	2	3	2	5	2	3	8	3	4	5	6	13
$\alpha(x)$	0	1	2	3	2	5	2	3	8	3	2	5	2	13

II.2. The sets $\mathcal{N}_\ell$

Definition. For each integer $\ell \geq 1$, we set

$$\mathcal{N}_\ell = \{x \in \mathbb{N}\,;\, \alpha(x) \geq F_\ell\}.$$

Set $F_0 = 1$. For $\ell \geq 2$, we find

$$\mathcal{N}_\ell = \{F_\ell,\ F_{\ell+1},\ F_{\ell+1}+F_{\ell-1},\ F_{\ell+2},\ F_{\ell+2}+F_{\ell-2},\ F_{\ell+2}+F_\ell,\ F_{\ell+3}, \dots\}$$

Theorem

Let $\ell \geq 3$ and write $\mathcal{N}_\ell = \{x_1 < x_2 < x_3 < \dots\}$.

- *Then $x_1 = F_\ell$ and the sequence of differences*

$$(x_2-x_1,\ x_3-x_2,\ x_4-x_3, \dots) = (F_{\ell-1},\ F_{\ell-1},\ F_{\ell-2},\ F_{\ell-2},\ F_{\ell-1},\ F_{\ell-1}, \dots)$$

  is the infinite Fibonacci word on $(F_{\ell-1},\ F_{\ell-1})$ and $(F_{\ell-2},\ F_{\ell-2})$.

- *For each $i \geq 1$, we have $\alpha(x_i) = F_\ell$ if and only if $i = 1$ or $i > 1$ and $x_i - x_{i-1} \neq x_{i+1} - x_i$.*

II.3. Example ($\ell = 3$):

$$\mathcal{N}_3 = \{x \in \mathbb{N}\,;\, \alpha(x) \geq F_3 = 3\} = \{x_1 < x_2 < x_3 < \cdots\}$$

with $x_1 = 3$ and

$$(x_{i+1} - x_i)_{i\geq 1} = (2, 2, 1, 1, 2, 2, 2, 2, 1, 1, 2, 2, 1, 1, 2, 2, \dots)$$

$$= \text{image of } w_\infty = abaababa\dots \text{ by } a \mapsto (2,2),\ b \mapsto (1,1).$$

We find:

$\alpha(x)$	3	5	3	8	3		5	13
x	3	5	7	8	9		11	13
differences		2	2	1	1	$\neq$	2	2

Going further, we obtain

$x \in \mathcal{N}_3$	3	5	7	8	9	11	13	15	17	18	19	21	23	24	26
diff.		2	2	1	1	2	2	2	2	1	1	2	2	1	1
$x \in \mathcal{N}_4$		5		8		11	13	15		18		21		24	
diff.				3		3	2	2		3		3		3	
$x \in \mathcal{N}_5$				8			13			18		21		24	

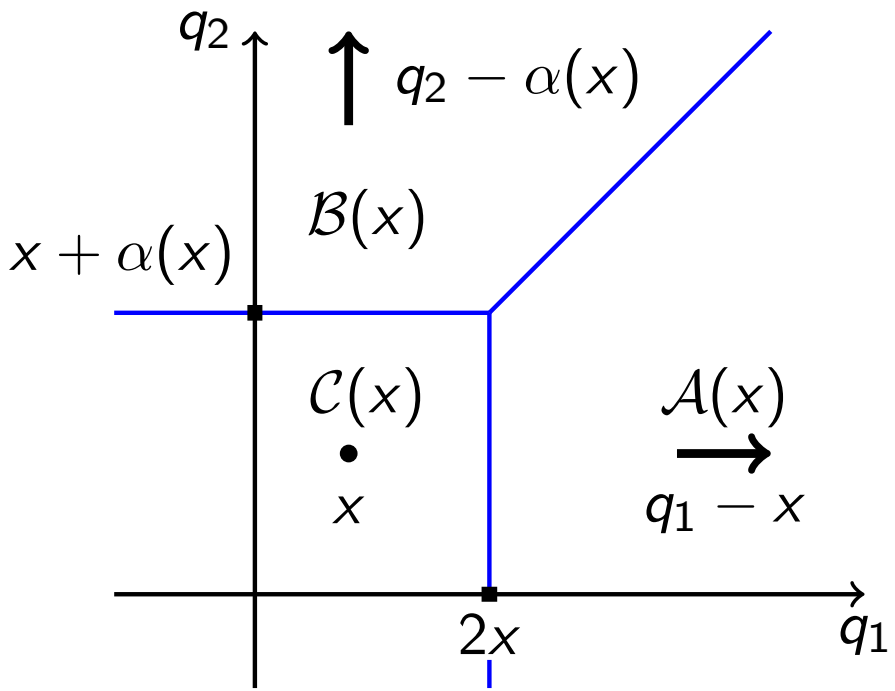
II.4. The ideal trajectories P_x

For each $x \in \mathbb{N}$, we define $P_x : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$P_x(q_1, q_2) = \Phi(q_1 - x, q_2 - \alpha(x), x)$$

We set

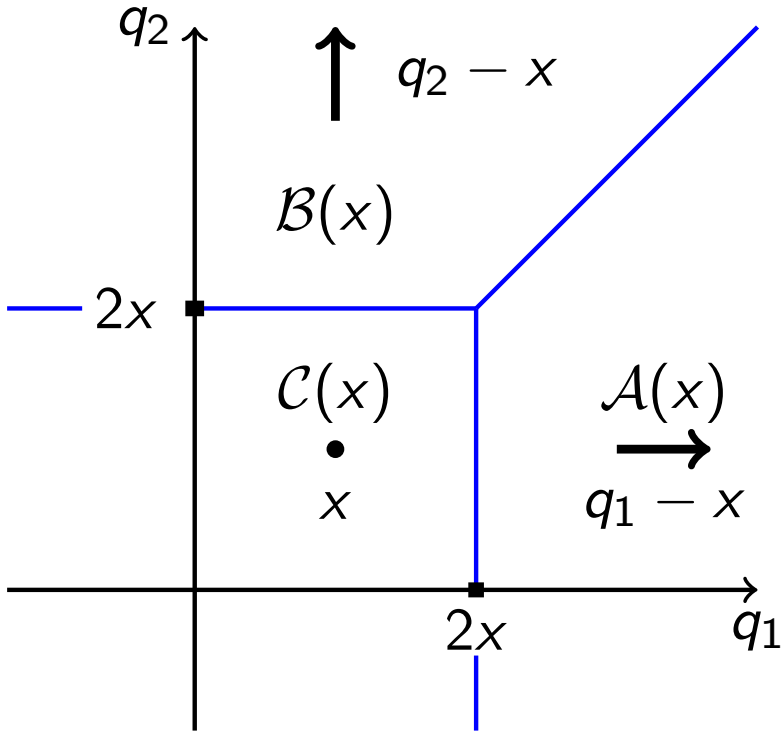
- $\mathcal{A}(x) = \{\mathbf{q} \in \mathbb{R}^2 \,;\, P_x(\mathbf{q}) = q_1 - x\}$,
- $\mathcal{B}(x) = \{\mathbf{q} \in \mathbb{R}^2 \,;\, P_x(\mathbf{q}) = q_2 - \alpha(x)\}$,
- $\mathcal{C}(x) = \{\mathbf{q} \in \mathbb{R}^2 \,;\, P_x(\mathbf{q}) = x\}$.



For example, if $x \in \mathcal{F} = \{0, 1, 2, 3, 5, 8, \ldots,$ then $\alpha(x) = x$, so

$$P_x(q_1, q_2) = \Phi(q_1 - x, q_2 - x, x)$$

We find



In particular,

$$\mathcal{A}(0) = \{(q_1, q_2) \in \mathbb{R}^2 \,;\, q_1 \geq \max\{0, q_2\}\} \subseteq \bigcup_{x \in \mathcal{F}} \mathcal{C}(x) = \mathbb{R}^2.$$

II.5. The 3-system $\mathbf{P}$

We define a map $\mathbf{P} = (P_1, P_2, P_3)\colon \mathcal{A}(0) \to \mathbb{R}^3$ by

$$\mathbf{P}(\mathbf{q}) = \Phi(q_1 - x, q_2 + x - z, z)$$

where

- $x \in \mathbb{N}$ is largest such that $\mathbf{q} \in \mathcal{A}(x)$,
- $z \in \mathbb{N}$ is smallest with $z > x$ such that $\mathbf{q} \in \mathcal{C}(z)$.

Theorem

- *For each $i = 1, 2, 3$, the i-th component P_i is continuous on $A(0)$ and differentiable with gradient $(0, 0)$, $(1, 0)$ or $(0, 1)$ at each interior point $\mathbf{q} = (q_1, q_2)$ of $\mathcal{A}(0)$ with $q_1 \notin \mathbb{Z}$, $q_2 \notin \mathbb{Z}$ and $q_1 - q_2 \notin \mathbb{Z}$.*
- *$P_1(\mathbf{q}) \leq P_2(\mathbf{q}) \leq P_3(\mathbf{q})$ and $\mathbf{P}_1(\mathbf{q}) + \mathbf{P}_2(\mathbf{q}) + \mathbf{P}_3(\mathbf{q}) = q_1 + q_2$ at each $\mathbf{q} = (q_1, q_2) \in \mathcal{A}(0)$.*
- *If P_1 (resp. $P_1 + P_2$) is not locally convex at an interior point of $\mathcal{A}(0)$, then $P_1(\mathbf{q}) = P_2(\mathbf{q})$ (resp. $P_2(\mathbf{q}) = P_3(\mathbf{q})$).*

We say that $\mathbf{P}$ is a rigid integral (2-parameter) 3-system.

II.6. Cellular decomposition

- For each $y \in \mathbb{N} \setminus \mathcal{F}$, we have $\alpha(y) = F_\ell$ for an integer $\ell \geq 2$ and there are unique $x, z \in \mathcal{N}_\ell$ such that $x < y < z$ are consecutive in $\mathcal{N}_\ell$. Then

$$\mathsf{Cell}(y) := \mathcal{A}(x) \cap \mathcal{B}(y) \cap \mathcal{C}(z)$$

is a compact polygon with 4 or 5 sides with non-empty interior on which

$$\mathbf{P}(\mathbf{q}) = \Phi(q_1 - x, q_2 - \alpha(y), z) = \Phi(P_x(\mathbf{q}), P_y(\mathbf{q}), P_{\mathbf{z}}(\mathbf{q})).$$

- For each $x \geq 0$,

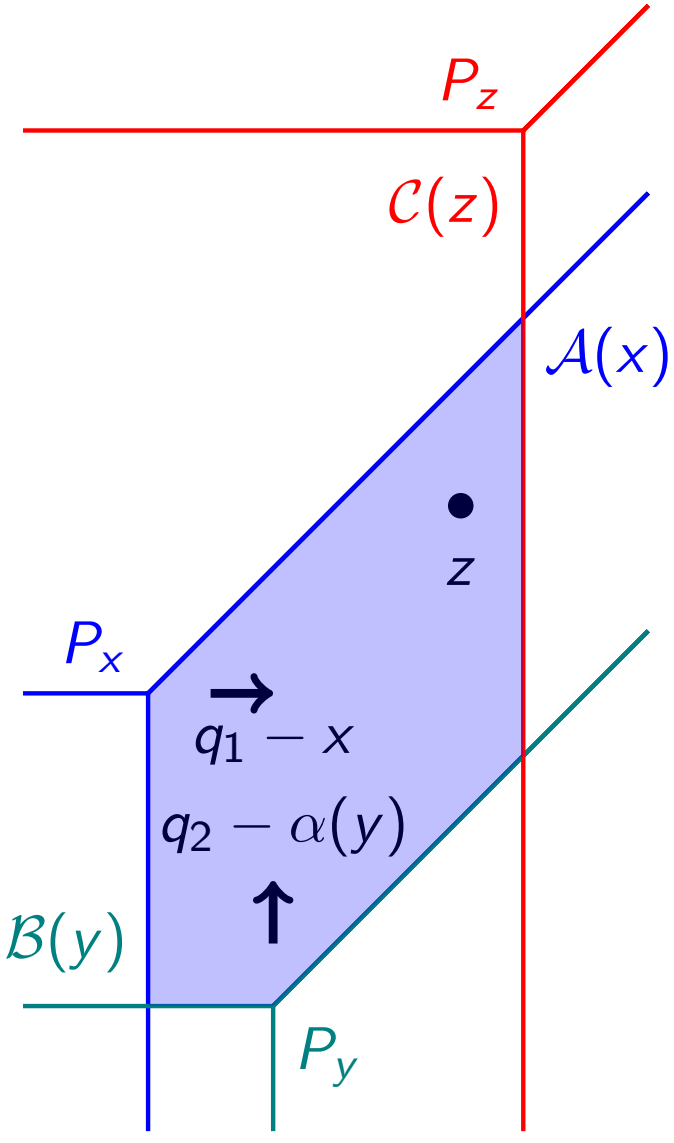
$$\mathsf{Trapeze}(x, x+1) := \mathcal{A}(x) \cap \mathcal{C}(x+1)$$

is an unbounded trapeze with two vertical sides and a top side of slope 0 or 1 on which

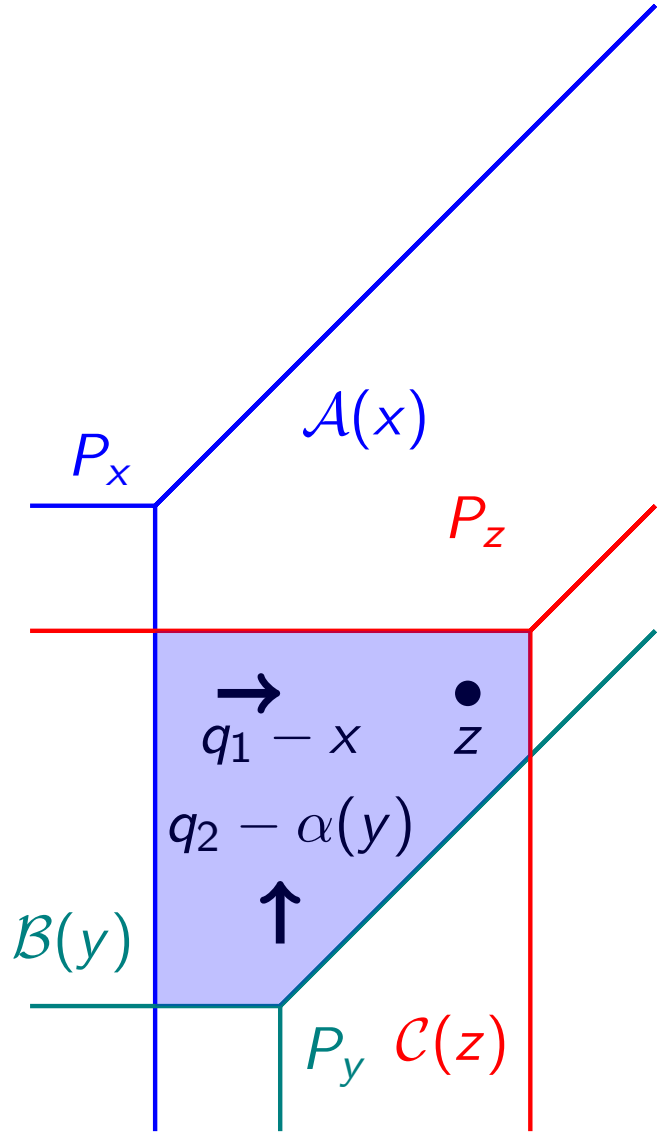
$$\mathbf{P}(\mathbf{q}) = \Phi(q_1 - x, q_2 - 1, x+1) = \Phi(P_x(\mathbf{q}), q_2 - 1, P_{x+1}(\mathbf{q})).$$

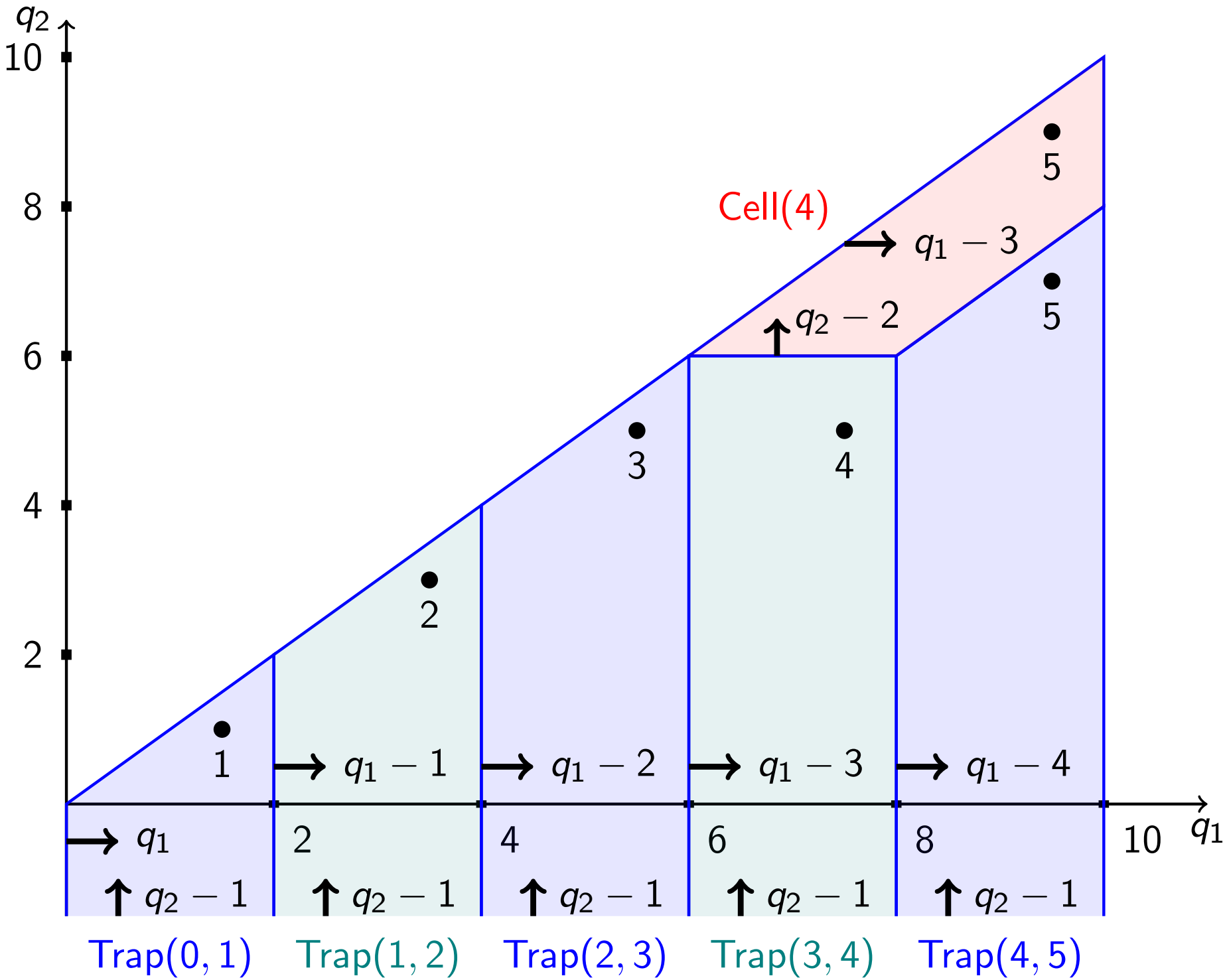
- These polygons have no interior point in common and cover $\mathcal{A}(0)$.

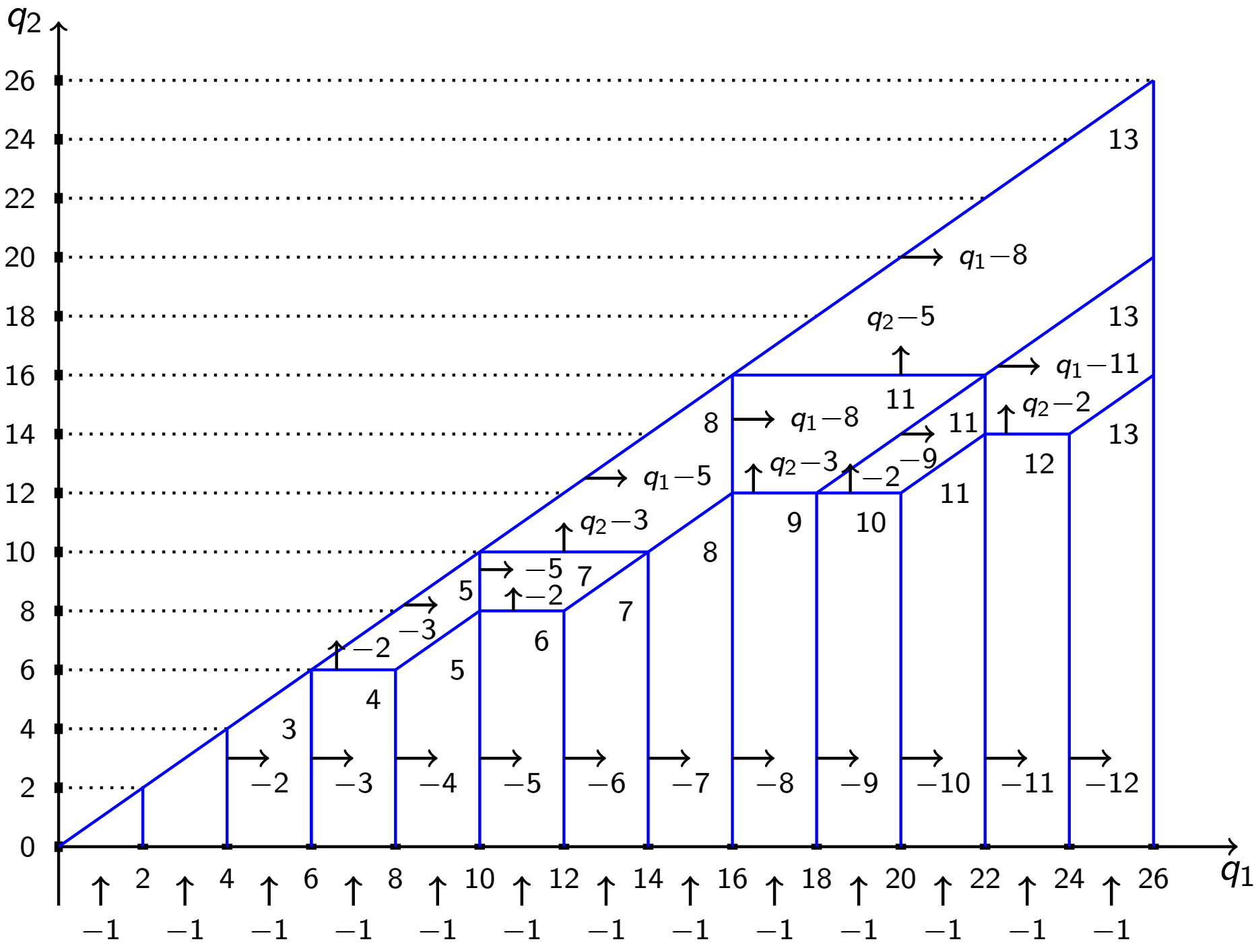
Typical cell : $\mathrm{Cell}(y) = \mathcal{A}(x) \cap \mathcal{B}(y) \cap \mathcal{C}(z)$



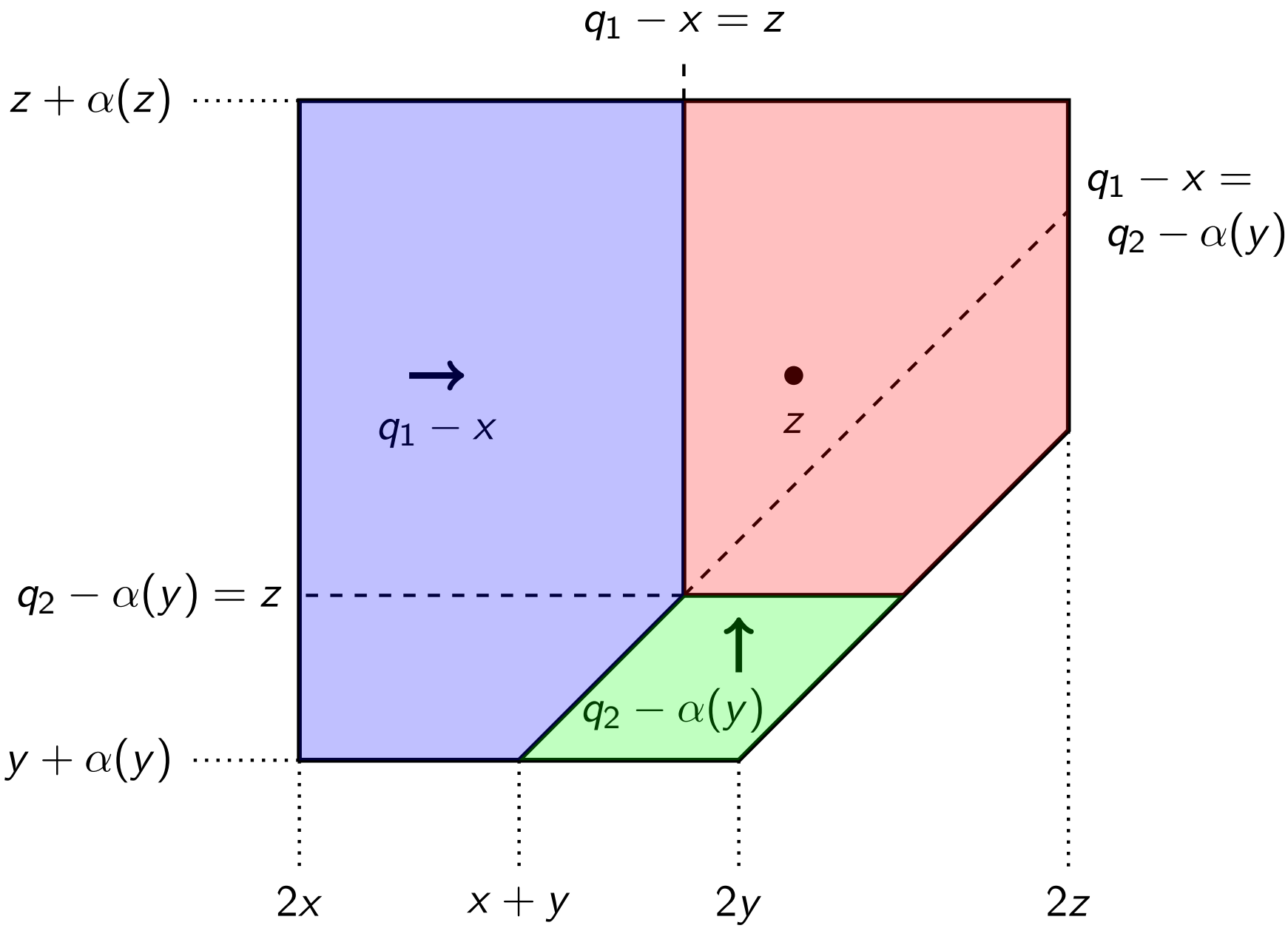
or




q_2
10
8
6
4
2
q_1
2
4
6
8
10
Cell(4)
$\rightarrow q_1 - 3$
$\uparrow q_2 - 2$
1
2
3
4
5
5
$\rightarrow q_1$
$\uparrow q_2 - 1$
$\rightarrow q_1 - 1$
$\uparrow q_2 - 1$
$\rightarrow q_1 - 2$
$\uparrow q_2 - 1$
$\rightarrow q_1 - 3$
$\uparrow q_2 - 1$
$\rightarrow q_1 - 4$
$\uparrow q_2 - 1$
Trap(0, 1)
Trap(1, 2)
Trap(2, 3)
Trap(3, 4)
Trap(4, 5)

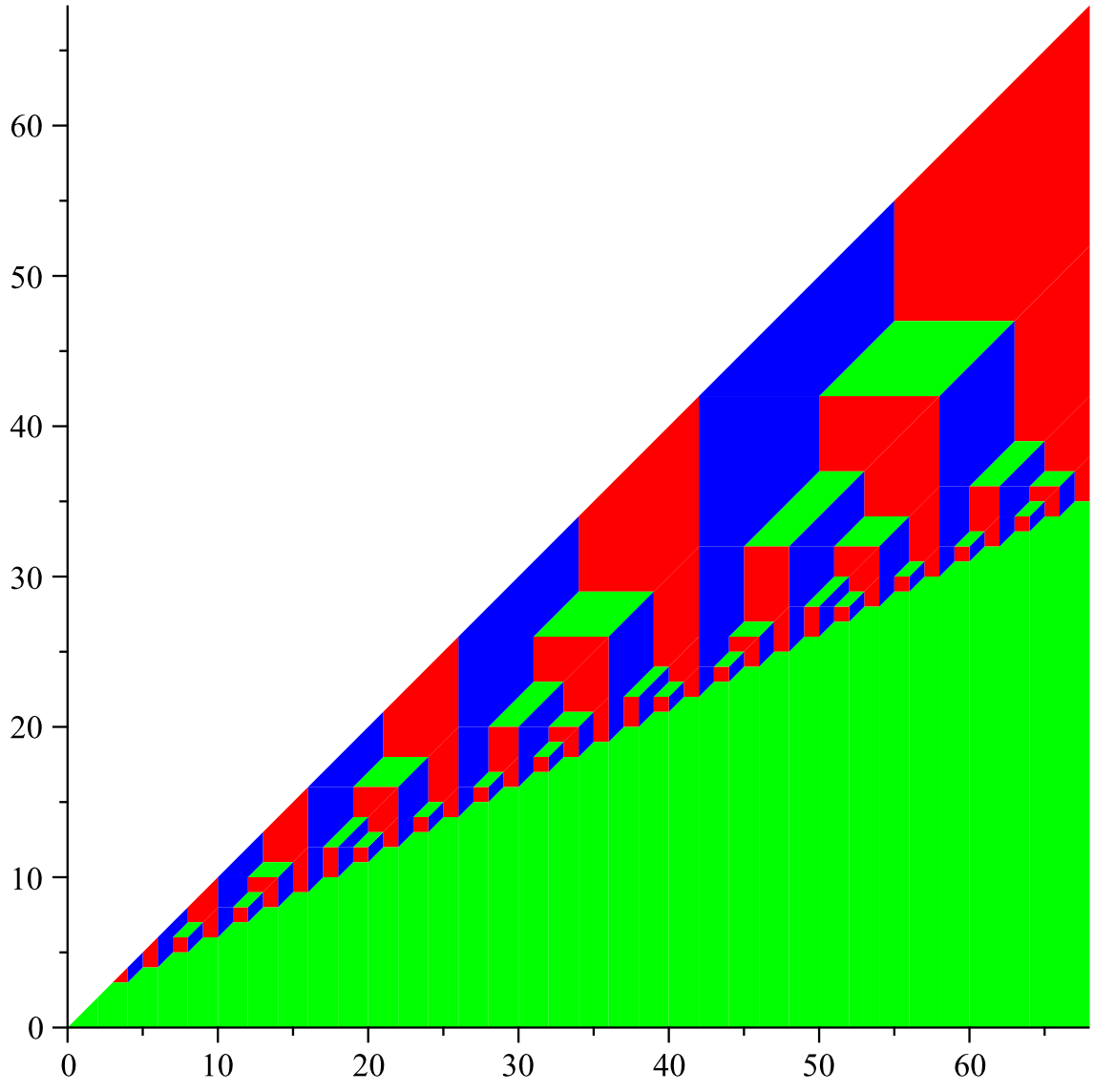

q_2
q_1
26
24
22
20
18
16
14
12
10
8
6
4
2
0
2
4
6
8
10
12
14
16
18
20
22
24
26
-1
13
q_1-8
q_2-5
11
q_1-11
q_2-2
12
q_1-5
q_2-3
-5
7
-2
-3
-2
-9
-2
-2
-3
-4
-5
-6
-7
-8
-9
-10
-11
-12

II.7. Graph of P_1 inside a cell



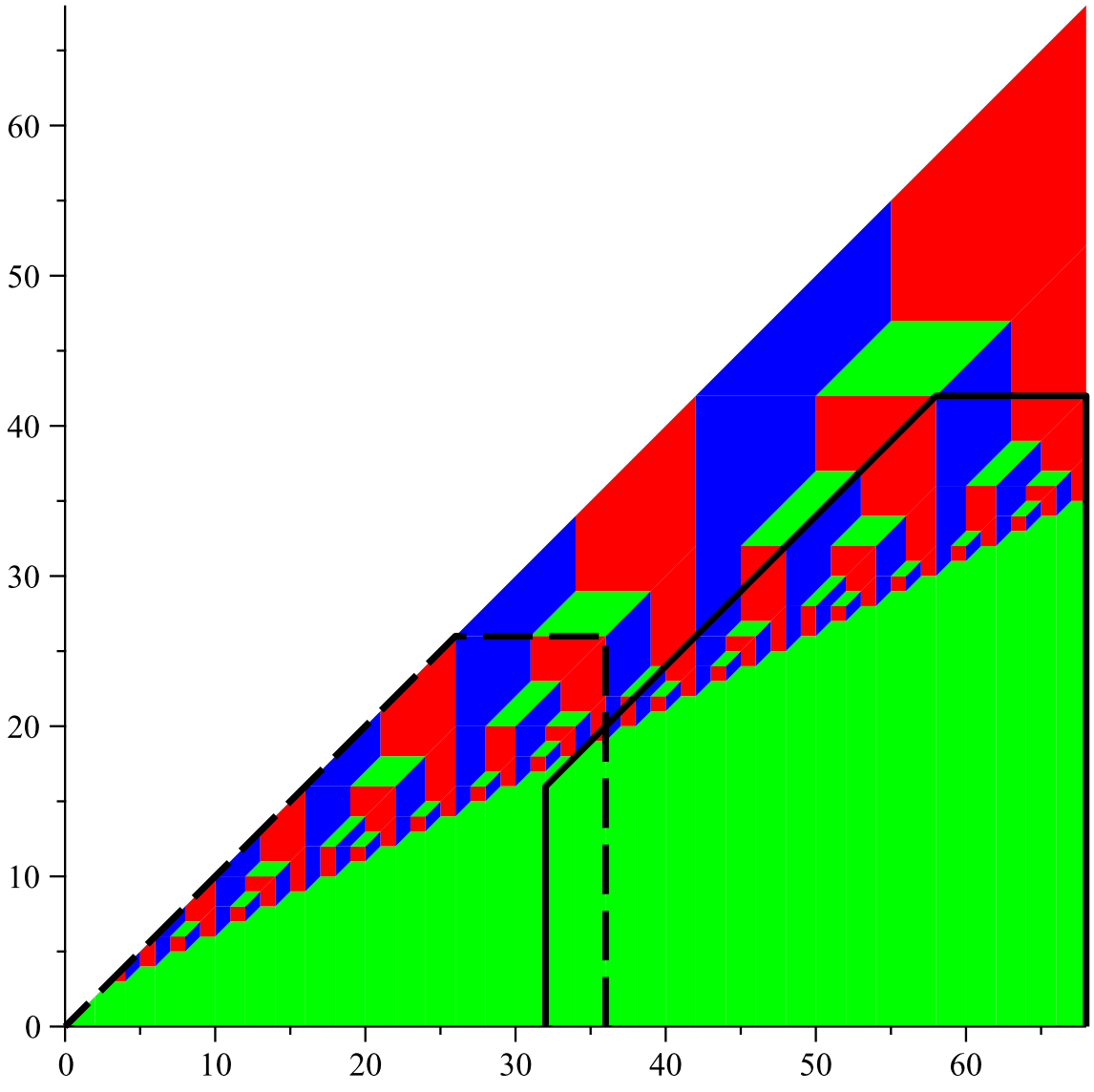
Graph of P_1

blue: gradient $(1, 0)$, green: gradient $(0, 1)$, red: gradient $(0, 0)$.



Graph of P_1

blue: gradient $(1, 0)$, green: gradient $(0, 1)$, red: gradient $(0, 0)$.



II.8. Properties

Theorem

- *For each integer* $k \geq 3$, *each* (q_1, q_2) *with*

$$0 \leq q_1 \leq 2F_{k-1} + 2F_{k-3} \quad \textit{and} \quad 0 \leq q_2 \leq \min\{q_1, 2F_{k-1}\},$$

and each $j = 1, 2, 3$, *we have*

$$P_j(q_1 + 4F_{k-2}, q_2 + 2F_{k-2}) = P_j(q_1, q_2) + 2F_{k-2}.$$

- *For each* $\mathbf{q} = (q_1, q_2)$ *with* $0 \leq q_1$ *and* $q_2 \leq q_1$, *we have*

$$\mathbf{P}(\gamma\mathbf{q}) = \gamma\mathbf{P}(\mathbf{q}) + \mathcal{O}(1).$$

II.9. Main result

Theorem

Let ξ be the particular extremal real number defined in Part I. There exists $\rho > 0$ such that the point $\boldsymbol{\xi} = (1, \xi, \xi^2)$ satisfies

$$\mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q}) = \rho \mathbf{P}(\rho^{-1}\mathbf{q}) + \mathcal{O}(1)$$

for each $\mathbf{q} = (q_1, q_2) \in \mathcal{A}(0)$.

Part III. The points

III.1. The Fibonacci sequence in $\mathrm{SL}_2(\mathbb{Z})$ attached to ξ

Recall that the sequence of symmetric matrices $(\mathbf{x}_i)_{i\geq 1}$ in $\mathrm{SL}_2(\mathbb{Z})$ attached to ξ is given by

$$\mathbf{x}_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \ \mathbf{x}_2 = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}, \ \mathbf{x}_{i+2} = \mathbf{x}_{i+1} M_{i+1} \mathbf{x}_i \quad (i \geq 1)$$

where

$$M_i = M = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix} \text{ if } i \text{ is even, and } M_i = {}^tM = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} \text{ else.}$$

Set $W_i = x_i M_i$ for each $i \geq 1$. We find

$$W_1 = \begin{pmatrix} 7 & -2 \\ 4 & -1 \end{pmatrix}, \ W_2 = \begin{pmatrix} 12 & 5 \\ 7 & 3 \end{pmatrix}, \ W_{i+2} = W_{i+1} W_i \quad (i \geq 1).$$

So $(W_i)_{i\geq 1}$ is a Fibonacci sequence in $\mathrm{SL}_2(\mathbb{Z})$. We extend it to a Fibonacci sequence $(W_i)_{i\geq 0}$ starting with

$$W_0 = W_2 W_1^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

III.2. The morphism associated to ξ

We denote by $\varphi\colon \{a,b\}^* \to \mathrm{SL}_2(Z)$ the morphism of monoids with

$$\varphi(a) = W_1 = \begin{pmatrix} 7 & -2 \\ 4 & -1 \end{pmatrix} \quad \text{and} \quad \varphi(b) = W_0 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then

$$\varphi(w_i) = W_i \quad (i \geq 1),$$

where $(w_i)_{i\geq 1} = (a,\ ab,\ aba, ababa, \dots)$.

For each non-empty word v in $\{a,b\}^*$, we set

$$M(v) = \begin{cases} M & \text{if } v \text{ ends in b,} \\ {}^tM & \text{if } v \text{ ends in a.} \end{cases}$$

For each $i \geq 1$, we have $M(w_i) = M_i$ and so

$$\varphi(w_i)M(w_i)^{-1} = W_i M_i^{-1} = \mathbf{x}_i$$

is a symmetric matrix.

III.3. The points $\mathbf{x}(v)$

For each non-empty word v in $\{a, b\}^*$, we denote by

$$\mathbf{x}(v) = \begin{pmatrix} x_0(v) & x_1(v) \\ x_1(v) & x_2(v) \end{pmatrix}$$

the symmetric matrix in $\mathrm{Mat}_{2\times 2}(\mathbb{Z})$ which has the same first column as $\varphi(v)M(v)^{-1}$ and satisfies

$$|x_1(v)\xi - x_2(v)| < 1/2.$$

This specifies $x_2(v)$ uniquely as $\xi \notin \mathbb{Q}$.

For the first non-empty prefixes v of $w_\infty = abaababa\ldots$, we find

$$\mathbf{x}(a) = \mathbf{x}_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \mathbf{x}(ab) = \mathbf{x}_2 = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix},$$

$$\mathbf{x}(aba) = \mathbf{x}_3 = \begin{pmatrix} 29 & 17 \\ 17 & 10 \end{pmatrix}, \quad \mathbf{x}(abaa) = \begin{pmatrix} 179 & 105 \\ 105 & 62 \end{pmatrix}.$$

III.4. Recurrence relations

For each integer $\ell \geq 1$, we set $\mathcal{W}_\ell = \{v \in]\epsilon, w_\infty[\,; |v| \in \mathcal{N}_\ell\}$.

For $\ell \geq 3$, we have

$$\mathcal{N}_\ell = \{F_\ell, F_{\ell+1}, F_{\ell+1} + F_{\ell-1}, F_{\ell+2}, F_{\ell+2} + F_{\ell-2}, \dots\},$$

thus

$$\mathcal{W}_\ell = \{w_\ell, w_{\ell+1}, w_{\ell+1}w_{\ell-1}, w_{\ell+2}, w_{\ell+2}w_{\ell-2}, \dots\},$$

Proposition

If ℓ is large enough, we have

(1) $\mathbf{x}(w_\ell) = \mathbf{x}_\ell$,

(2) $\mathbf{x}(w_{\ell+1}w_{\ell-1}) = t_{\ell-1}\mathbf{x}_{\ell+1} + \mathbf{x}_\ell$,

(3) $\mathbf{x}(w_{\ell+2}w_{\ell-2}) = t_{\ell-2}\mathbf{x}_{\ell+2} + \mathbf{x}(w_{\ell+1}w_{\ell-1})$,

(4) $\mathbf{x}_{\ell+2} = t_\ell \mathbf{x}_{\ell+1} - \mathbf{x}_{\ell-1}$.

where $t_k := \mathrm{trace}(w_k) = 3x_{k,0} = 3\|\mathbf{x}_k\|$ for each $k \geq 1$.

III.5. Proof of (1) and (2)

Notation and preliminary observations:

- Write $A \equiv B$ when $A, B \in \mathrm{Mat}_{2\times 2}(\mathbb{Z})$ have the same first column.
- Then $AM^{-1} \equiv -A\,{}^tM^{-1}$ for each $A \in \mathrm{Mat}_{2\times 2}(\mathbb{Z})$ (specific to M).
- If $\mathbf{x}(v) \equiv A$ with A symmetric and $\|(\xi,-1)A\| < 1/2$, then $\mathbf{x}(v) = A$.

1. $\mathbf{x}(w_\ell) = \mathbf{x}_\ell$ if $\ell \gg 1$ because

$$\mathbf{x}(w_\ell) \equiv W_\ell M_\ell^{-1} = \mathbf{x}_\ell \quad \text{and} \quad \|(\xi,-1)\mathbf{x}_\ell\| \asymp \|\mathbf{x}_\ell\|^{-1}.$$

2. We have $w_{\ell+1}w_{\ell-1} = w_\ell w_{\ell-1}^2$, thus

$$\begin{aligned}
\varphi(w_{\ell+1}w_{\ell-1}) &= W_\ell W_{\ell-1}^2 \\
&= W_\ell\big(t_{\ell-1}W_{\ell-1} - I\big) \qquad \text{by Cayley-Hamilton Theorem,} \\
&= t_{\ell-1}W_{\ell+1} - W_\ell, \\
\Rightarrow\ \mathbf{x}(w_{\ell+1}w_{\ell-1}) &\equiv \varphi(w_{\ell+1}w_{\ell-1})M_{\ell-1}^{-1} \\
&= t_{\ell-1}W_{\ell+1}M_{\ell+1}^{-1} - W_\ell\,{}^tM_\ell^{-1} \equiv t_{\ell-1}\mathbf{x}_{\ell+1} + \mathbf{x}_\ell \quad \text{symmetric !}
\end{aligned}$$

and $\|(\xi,-1)(t_{\ell-1}\mathbf{x}_{\ell+1} + \mathbf{x}_\ell)\| \ll \|\mathbf{x}_\ell\|^{-1}$, thus

$$\mathbf{x}(w_{\ell+1}w_{\ell-1}) = t_{\ell-1}\mathbf{x}_{\ell+1} + \mathbf{x}_\ell \quad \text{if } \ell \gg 1.$$

III.6. Proof of (3) and (4)

3. Similarly,

$$w_{\ell+2}w_{\ell-2} = w_{\ell+1}w_\ell w_{\ell-2} = \big(w_{\ell+1}w_{\ell-1}\big)w_{\ell-2}^2$$

yields, when $\ell \gg 1$,

$$\mathbf{x}\big(w_{\ell+2}w_{\ell-2}\big) = t_{\ell-2}\mathbf{x}_{\ell+2} + \mathbf{x}\big(w_{\ell+1}w_{\ell-1}\big).$$

4. We also have

$$\mathbf{x}_{\ell+2} = \mathbf{x}_{\ell+1}M_{\ell+1}\mathbf{x}_\ell = (\mathbf{x}_{\ell-1}M_{\ell-1}\mathbf{x}_\ell)M_{\ell+1}\mathbf{x}_\ell = \mathbf{x}_{\ell-1}{}^t W_\ell^2,$$

which yields

$$\mathbf{x}_{\ell+2} = t_\ell \mathbf{x}_{\ell+1} - \mathbf{x}_\ell.$$

III.7. Initial palindromes in w_∞

$w_1 = a$, $w_2 = ab$, $w_3 = aba$, $w_4 = abaab$, $w_5 = abaababa$,...

- For each integer $i \geq 2$, we have

$$w_i = \begin{cases} p_i ab & \text{if } i \text{ is even,} \\ p_i ba & \text{if } i \text{ is odd.} \end{cases}$$

  for a palindrome $p_i \in \{a, b\}^*$ [Berstel].

- Thus, writing

$$\widetilde{w}_i = \begin{cases} p_i ba & \text{if } i \text{ is even,} \\ p_i ab & \text{if } i \text{ is odd,} \end{cases} \quad (i \geq 2),$$

  we find

$$w_{i+2} = w_{i+1} w_i = w_i \widetilde{w}_{i+1} \quad (i \geq 1).$$

III.8. Consecutive words of $\mathcal{W}_\ell$

Recall, for $\ell \geq 3$, consecutive numbers in $\mathcal{N}_\ell$ differ by $F_{\ell-2}$ or $F_{\ell-1}$ and that

$$\mathcal{W}_\ell = \{v \in]\epsilon, w_\infty[\,;\, |v| \in \mathcal{N}_\ell\}.$$

Proposition

Suppose that $\ell \geq 4$ and write

$$\mathcal{W}_\ell = \{v_1 < v_2 < v_3 < \dots\}.$$

Then, we have $v_1 = w_\ell$ and, for each $i \geq 1$,

$$v_{i+1} = v_i u_i \quad \text{with} \quad u_i \in \{w_{\ell-2},\ \widetilde{w}_{\ell-2},\ w_{\ell-1},\ \widetilde{w}_{\ell-1}\}.$$

Illustration on the first 12 elements of $\mathcal{W}_\ell$

Suppose that $\ell \geq 4$ and write $\mathcal{W}_\ell = \{v_1 < v_2 < v_3 < \dots\}$, then

$$
\begin{aligned}
v_1 &= w_\ell, \\
v_2 &= w_{\ell+1} && = v_1 w_{\ell-1}, \\
v_3 &= w_{\ell+1} w_{\ell-1} && = v_2 w_{\ell-1}, \\
v_4 &= w_{\ell+2} && = v_3 w_{\ell-2}, \\
v_5 &= w_{\ell+2} w_{\ell-2} && = v_4 w_{\ell-2}, \\
v_6 &= w_{\ell+2} w_\ell && = v_5 \widetilde{w}_{\ell-1}, \\
v_7 &= w_{\ell+3} && = v_6 w_{\ell-1}, \\
v_8 &= w_{\ell+3} w_{\ell-1} && = v_7 w_{\ell-1}, \\
v_9 &= w_{\ell+3} w_\ell w_{\ell-3} && = v_8 w_{\ell-1}, \\
v_{10} &= w_{\ell+3} w_{\ell+1} && = v_9 \widetilde{w}_{\ell-2}, \\
v_{11} &= w_{\ell+3} w_{\ell+1} w_{\ell-2} && = v_{10} w_{\ell-2}, \\
v_{12} &= w_{\ell+4} && = v_{11} \widetilde{w}_{\ell-1}, \dots
\end{aligned}
$$

III.9. General recurrence relation

Theorem

There exists $\ell_0 \geq 4$ such that, for any $\ell \geq \ell_0$ and any choice of consecutive words $u < v < w$ of $\mathcal{W}_\ell$, the points $\mathbf{x}(u)$, $\mathbf{x}(v)$, $\mathbf{x}(w)$ are linearly independent if and only if $v \notin \mathcal{W}_{\ell+1}$. More precisely,

(i) *if $v \in \mathcal{W}_{\ell+1}$, then $|v| - |u| = |w| - |v| = F_k$ with $k \in \{\ell - 2, \ell - 1\}$ and*

$$\mathbf{x}(w) = t_k \mathbf{x}(v) \pm \mathbf{x}(u);$$

(ii) *if $v \in \mathcal{W}_{\ell+1}$, then* $\det(\mathbf{x}(u), \mathbf{x}(v), \mathbf{x}(w)) = \pm 2$.

In case (i), the word w ends in w_k^2, $\widetilde{w}_k^2$, $w_k \widetilde{w}_k$ or $\widetilde{w}_k w_k$, and we use the fact that, for any word $m \in \{a, b\}^*$, we have

$$\mathbf{x}(mab) = \mathbf{x}(mab).$$

Thus, $\mathbf{x}(w) = \mathbf{x}(ms^2)$ where $m \leq w$ and $s \in \{w_k, \widetilde{w}_k\}$.

III.10. The trajectory of $\mathbf{x}(v)$

Proposition

There exists $\rho > 0$ such that, for any $v \in]\epsilon, w_\infty[$, we have

(i) $\log x_0(v) = \rho|v| + \mathcal{O}(1)$,

(ii) $\log |x_0(v)\xi - x_1(v)| = -\rho|v| + \mathcal{O}(1)$,

(iii) $\log |x_0(v)\xi^2 - x_2(v)| = -\rho\alpha(|v|) + \mathcal{O}(1)$,

and so

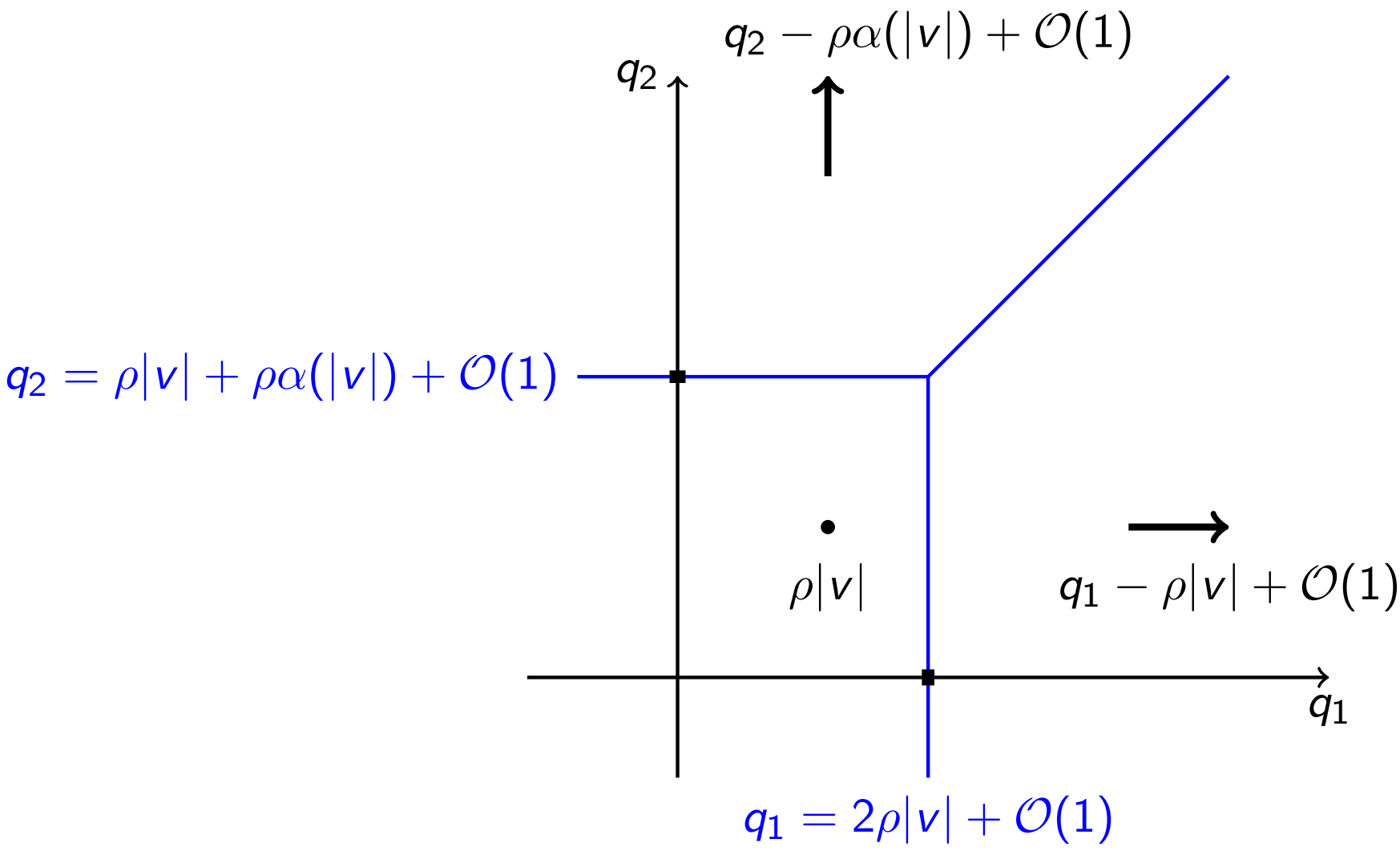
$$L_{\mathbf{x}(v)}(\mathbf{q}) = \rho P_{|v|}(\rho^{-1}\mathbf{q}) + \mathcal{O}(1) \quad (\mathbf{q} \in \mathbb{R}^2).$$

Sketch of proof.

- We first show the existence of ρ such that (i) holds.
- Then (ii) follows from the fact that $x_1(v)/x_0(v)$ is a convergent of ξ in reduced form, for each $v \in]\epsilon, w_\infty[$.
- The general recurrence relation yields (iii).



Graph of $L_{\mathbf{x}(v)}$



III.11. Proof of the main result

Theorem

We have $\mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q}) = \rho\mathbf{P}(\rho^{-1}\mathbf{q}) + \mathcal{O}_{\xi}(1)$ *for each* $\mathbf{q} \in \mathcal{A}(0)$.

Sketch of proof

- It suffices to show this on $\rho\mathsf{Cell}(y)$ for each $y \in \mathbb{N} \setminus \mathcal{F}$ with $\alpha(y) = F_\ell \geq F_{\ell_0}$.
- We have $\mathsf{Cell}(y) = \mathcal{A}(x) \cap \mathcal{B}(y) \cap \mathcal{C}(z)$ with $x < y < z$ consecutive in $\mathcal{N}_\ell$. Moreover, $z - x = F_\ell = \alpha(y)$ and

$$\mathbf{P}(\mathbf{q}) = \Phi(P_x(\mathbf{q}), P_y(\mathbf{q}), P_z(\mathbf{q})) \text{ for each } \mathbf{q} \in \mathsf{Cell}(y).$$

- Write $x = |u|$, $y = |v|$ and $z = |w|$ for consecutive words $u < v < w$ in $\mathcal{W}_\ell$. Since $v \notin \mathcal{W}_{\ell+1}$, the points $\mathbf{x}(u)$, $\mathbf{x}(v)$ and $\mathbf{x}(w)$ are linearly independent.
- For $\mathbf{q} = (q_1, q_2) \in \rho\mathsf{Cell}(y)$, we find

$$\begin{aligned}\left\|\rho\mathbf{P}(\rho^{-1}\mathbf{q}) - \mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q})\right\| &= \left\|\Phi(L_{\mathbf{x}(u)}(\mathbf{q}), L_{\mathbf{x}(v)}(\mathbf{q}), L_{\mathbf{x}(w)}(\mathbf{q})) - \mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q})\right\| + \mathcal{O}(1)\\ &\leq L_{\mathbf{x}(u)}(\mathbf{q}) + L_{\mathbf{x}(v)}(\mathbf{q}) + L_{\mathbf{x}(w)}(\mathbf{q}) - (q_1 + q_2) + \mathcal{O}(1)\\ &\leq (q_1 - \rho x) + (q_2 - \rho F_\ell) + \rho z - (q_1 + q_2) + \mathcal{O}(1) = \mathcal{O}(1).\end{aligned}$$

III.12. Final remark

Let $\theta\colon \{a,b\}^* \to \{a,b\}^*$ denote the morphism of monoids determined by $\theta(b) = a$ and $\theta(a) = ab$.

It satisfies

- $\theta(w_i) = w_{i+1}$ for each $i \geq 1$;
- $\theta(\mathcal{W}_\ell) = \mathcal{W}_{\ell+1}$ for each $\ell \geq 4$;
- $|\theta(v)| = \gamma|v| + \mathcal{O}(1)$ for each $v \in [\epsilon, w_\infty[$.

Thus, for each $\mathbf{q} \in \mathcal{A}(0)$,

$$\mathbf{P}(\gamma\mathbf{q}) = \gamma\mathbf{P}(\mathbf{q}) + \mathcal{O}(1) \quad \text{and} \quad \mathbf{L}_{\boldsymbol{\xi}}(\gamma\mathbf{q}) = \gamma\mathbf{L}_{\boldsymbol{\xi}}(\mathbf{q}) + \mathcal{O}(1).$$