

Diophantine approximation with sign constraints

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Schmidt's open problem

Dirichlet (1842). Let $\alpha, \beta \in \mathbb{R}$. Then

$$|x_0 + x_1\alpha + x_2\beta| \leq X^{-2}, \quad |x_1| \leq X, \quad |x_2| \leq X$$

admits a solution $\mathbf{x} = (x_0, x_1, x_2) \in \mathbb{Z}^3 \setminus \{0\}$ **for each** $X \geq 1$.

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Theorem A (Schmidt, Monat. Math. 1976)

Suppose that $1, \alpha, \beta$ are linearly independent over $\mathbb{Q}$. Let $\epsilon > 0$. Then, there are arbitrarily large X for which

$$|x_0 + x_1\alpha + x_2\beta| \leq \epsilon X^{-\gamma}, \quad 0 < x_1 \leq X, \quad 0 < x_2 \leq X$$

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- **Moshchevitin** (MJNTC, 2012): the exponent $\gamma = 1.618\dots$ cannot be replaced by a number $> 1.946\dots$

Schmidt's refined result

Theorem B (Schmidt)

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3 \setminus \{0\}$ with $\mathbf{u} \cdot \mathbf{v} = 0$. Suppose that $\mathbf{u}$ has $\mathbb{Q}$ -linearly independent coordinates. For any $\epsilon > 0$ and $\delta > 0$, there exist $\mathbf{x} \in \mathbb{Z}^3 \setminus \{0\}$ such that

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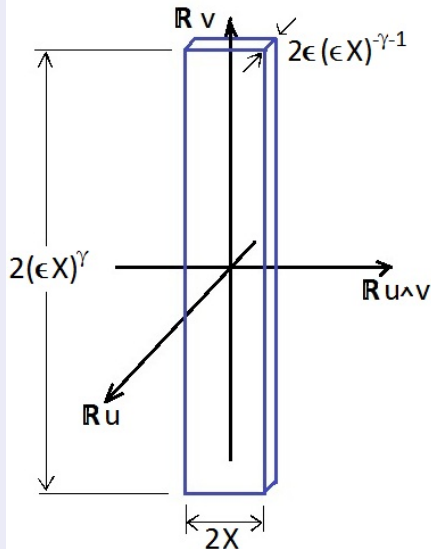
$$\angle(\mathbf{x}, \mathbf{v}) \leq \delta \quad \text{and} \quad |\mathbf{x} \cdot \mathbf{u}| \leq \epsilon \|\mathbf{x}\|^{-\gamma}.$$

- Theorem B $\Rightarrow$ Theorem A: Take $\mathbf{u} = (1, \alpha, \beta)$, $\mathbf{v} = (-\alpha - \beta, 1, 1)$, δ small enough, and let $\epsilon \rightarrow 0$.

Proof of Schmidt's result

By contradiction...

- $\text{vol}(\mathcal{C}_X) = 8$
 $\Rightarrow \mathcal{C}_X \cap \mathbb{Z}^3 \neq \{0\}$

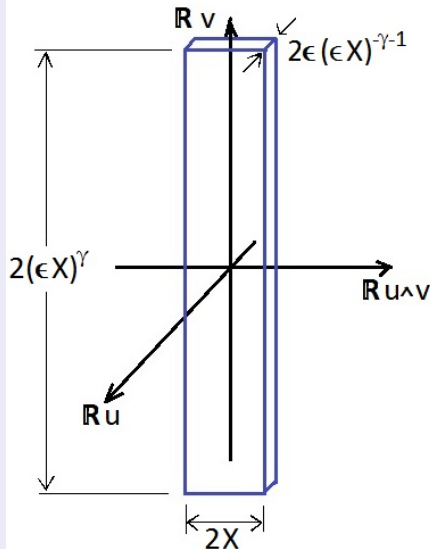


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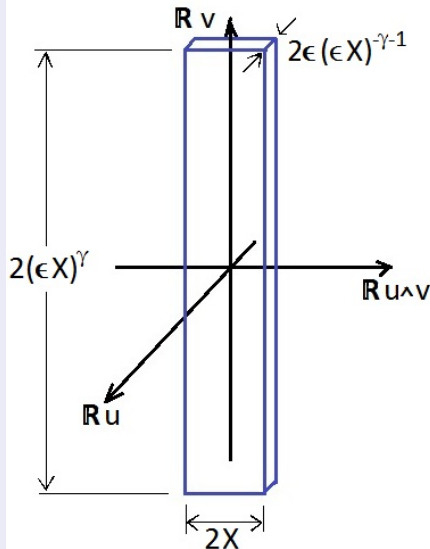
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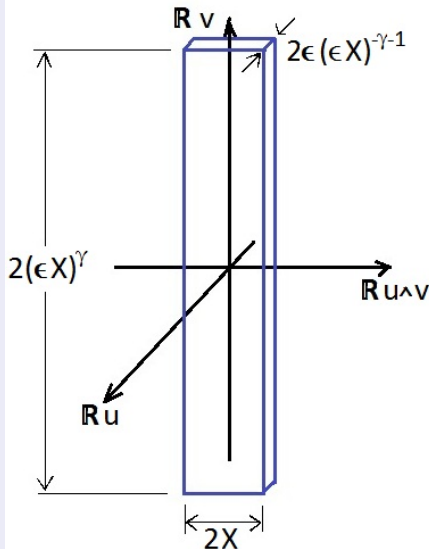
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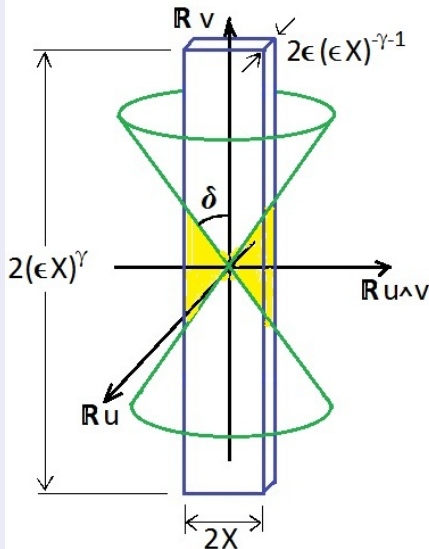
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- Thus $\angle(\mathbf{x}, \mathbf{v}) \geq \delta$

$$\Rightarrow \|\mathbf{x}\| \ll X$$



- Thus, for each $X \geq 1$, there exists $\mathbf{x} \in \mathbb{Z}^3 \setminus \{0\}$ such that

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- There exist $\mathbf{x}_1, \mathbf{x}_2, \dots$ in $\mathbb{Z}^3$ such that

- ▶ $X_i := \|\mathbf{x}_i\| \uparrow \infty$,
- ▶ $L_i := |\mathbf{x}_i \cdot \mathbf{u}| \downarrow 0$,
- ▶ if $\|\mathbf{x}\| < X_{i+1}$ for some $\mathbf{x} \in \mathbb{Z}^3$, then $|\mathbf{x} \cdot \mathbf{u}| > L_i$.

$$\Rightarrow \boxed{L_i = |\mathbf{x}_i \cdot \mathbf{u}| \ll X_{i+1}^{-\gamma-1}} \text{ for each } i.$$

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- W. Schmidt uses two more arguments to conclude...

Main result

Theorem C

Let $\psi: (1, \infty) \rightarrow (1, \infty)$ be increasing and unbounded.

Let $\mathbf{x}_0$ be a primitive point of $\mathbb{Z}^3$, and let $\delta > 0$.

Then, there exists $\mathbf{u} \in \mathbb{R}^3 \setminus \{0\}$ with the following properties.

- 1) $\mathbf{u}$ has $\mathbb{Q}$ -linearly independent coordinates;
- 2) any $\mathbf{x} \in \mathbb{Z}^3 \setminus \{0\}$ with $\angle(\mathbf{x}, \mathbf{x}_0) \geq \delta$ and $\|\mathbf{x}\|$ large enough satisfies

$$|\mathbf{x} \cdot \mathbf{u}| \geq \frac{1}{\psi(\|\mathbf{x}\|)\|\mathbf{x}\|^\gamma}.$$

Construction

Step 1. Let $\left(\frac{p_n}{q_n}\right)_{n \geq 1}$ be the sequence of convergents of a badly approximable real number α . (e.g. $\alpha = \sqrt{2} - 1$ or $(\sqrt{5} - 1)/2$)

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Then $\left| \frac{p}{q} - \frac{p_n}{q_n} \right| \gg \frac{1}{q^2}$ if $|q| < q_n$.

Step 2. Fix a sequence $(X_i)_{i \geq 0}$ with

$$X_0 = \|\mathbf{x}_0\| \quad \text{and} \quad X_{i+1} \geq X_i^\gamma \quad \text{for each} \quad i \geq 0.$$

Expand $\mathbf{x}_0$ to a basis $(\mathbf{x}_0, \mathbf{x}'_0, \mathbf{x}''_0)$ of $\mathbb{Z}^3$ with $\angle(\mathbf{x}_0, \mathbf{x}'_0) \leq \delta/8$, and put

$$\mathbf{x}_1 = \left\lfloor \frac{2X_1}{\|\mathbf{x}'_0\|} \right\rfloor \mathbf{x}'_0 + \mathbf{x}''_0 \quad \text{and} \quad \mathbf{y}_1 = \left\lfloor \frac{2X_1^\gamma}{\|\mathbf{x}_0\|} \right\rfloor \mathbf{x}_0 + \mathbf{x}'_0.$$

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Then $(\mathbf{x}_0, \mathbf{x}_1, \mathbf{y}_1)$ is a basis of $\mathbb{Z}^3$ since

$$\det(\mathbf{x}_0, \mathbf{x}_1, \mathbf{y}_1) = \det(\mathbf{x}_0, \mathbf{x}'_0, \mathbf{x}''_0) = \pm 1.$$

Suppose that $(\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{y}_i)$ has been constructed for some integer $i \geq 1$ and is a basis of $\mathbb{Z}^3$. Define

$$\mathbf{x}_{i+1} = q_n \mathbf{y}_i - p_n \mathbf{x}_{i-1} - a \mathbf{x}_i, \quad \mathbf{y}_{i+1} = q_{n+1} \mathbf{y}_i - p_{n+1} \mathbf{x}_{i-1} + b \mathbf{x}_i$$

where $n \geq 2$ is determined by $q_{n-1} \leq \frac{2X_{i+1}}{X_i^\gamma} < q_n$ and where

$$a = \left\lfloor \frac{q_n \det(\mathbf{x}_{i-1}, \mathbf{y}_i, \mathbf{x}_{i-1} \wedge \mathbf{x}_i)}{\|\mathbf{x}_{i-1} \wedge \mathbf{x}_i\|^2} \right\rfloor, \quad b = \left\lfloor \frac{2X_{i+1}^\gamma}{\|\mathbf{x}_i\|} \right\rfloor.$$

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Then, $(\mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{y}_{i+1})$ is again a basis of $\mathbb{Z}^3$ because

$$\det(\mathbf{x}_i, \mathbf{x}_{i+1}, \mathbf{y}_{i+1}) = (p_n q_{n+1} - q_n p_{n+1}) \det(\mathbf{x}_{i-1}, \mathbf{x}_i, \mathbf{y}_i) = \pm 1.$$

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For each $i \geq 1$, we set $\pi_i = \langle \mathbf{x}_{i-1}, \mathbf{x}_i \rangle_{\mathbb{R}}$ and we form the unit vector

$$\mathbf{u}_i = \frac{1}{\|\mathbf{x}_{i-1} \wedge \mathbf{x}_i\|} \mathbf{x}_{i-1} \wedge \mathbf{x}_i$$

perpendicular to the plane π_i .

Step 3. If X_1 is large enough, then for each $i \geq 1$, we have

$$\|\mathbf{x}_i\| \asymp X_i$$

and there exists a unit vector $\mathbf{u} \in \mathbb{R}^3$ with $\mathbb{Q}$ -linearly independent coordinates such that

$$\angle(\mathbf{u}_i, \mathbf{u}) \ll \frac{1}{X_{i-1}X_i^{\gamma+1}} \quad \text{and} \quad |\mathbf{x}_i \cdot \mathbf{u}| \asymp \frac{1}{X_{i+1}^{\gamma+1}}$$

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Corollary

For each $X \geq 1$, there exists $\mathbf{x} \in \mathbb{Z}^3 \setminus \{0\}$ such that

$$\|\mathbf{x}\| \leq X \quad \text{and} \quad |\mathbf{x} \cdot \mathbf{u}| \ll \frac{1}{X^{\gamma+1}}$$

Moreover if $X_{i+1} \geq X_{i-1}X_i^{\gamma+2}$ for each $i \geq 1$ and if X_1 is large enough, then there also exist unit vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ such that

$$\left. \begin{array}{ll} \text{if } i \text{ is odd} & \angle(\mathbf{x}_i, \mathbf{v}) \\ \text{if } i \text{ is even} & \angle(\mathbf{x}_i, \mathbf{w}) \end{array} \right\} \ll \frac{1}{X_i X_{i+1}^{\gamma+1}} = o\left(\frac{1}{X_{i+1}^{\gamma+1}}\right).$$

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$$\|\mathbf{x}\| \leq X, \quad |\mathbf{x} \cdot \mathbf{u}| \ll \frac{1}{X^{\gamma+1}}, \quad \min\{\angle(\mathbf{x}, \mathbf{v}), \angle(\mathbf{x}, \mathbf{w})\} = o\left(\frac{1}{X^{\gamma+1}}\right).$$

Moreover, $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ is a basis of $\mathbb{R}^3$ with

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w} = 0, \quad \angle(\mathbf{v}, \mathbf{x}_0) \leq \frac{\delta}{2}, \quad \angle(\mathbf{w}, \mathbf{x}_0) \leq \frac{\delta}{2}.$$

Finally, if $X_i \rightarrow \infty$ fast enough in connection with the choice of ψ , then each $\mathbf{x} \in \mathbb{Z}^3$ with $\|\mathbf{x}\|$ large enough satisfies

$$|\mathbf{x} \cdot \mathbf{u}| \geq \frac{\min\{\angle(\mathbf{x}, \mathbf{v}), \angle(\mathbf{x}, \mathbf{w})\}}{\psi(\|\mathbf{x}\|)\|\mathbf{x}\|^\gamma}.$$

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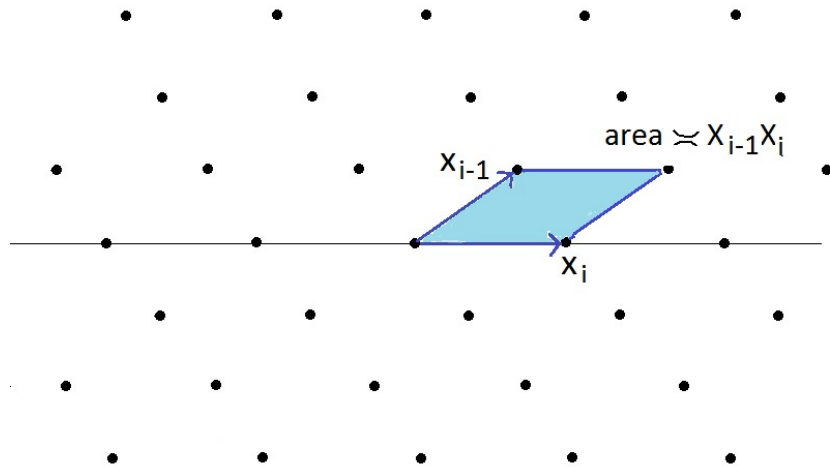
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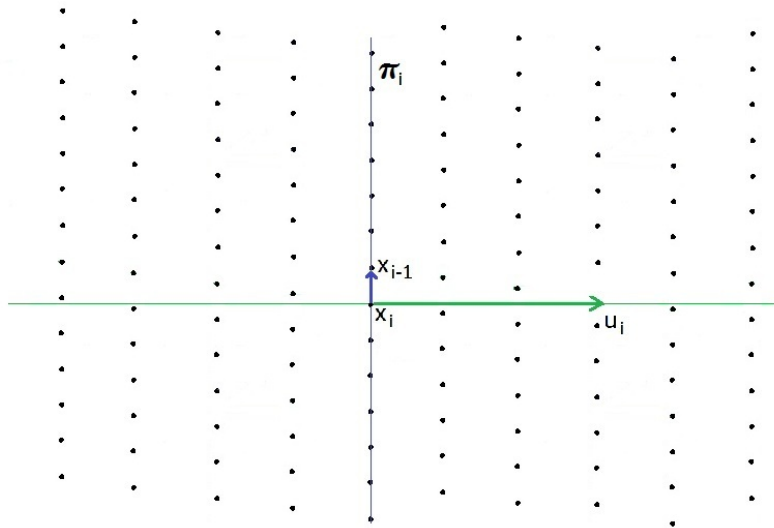
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If $\mathbf{x} \in \mathbb{Z}^3 \setminus \{0\}$ has $\angle(\mathbf{x}, \mathbf{x}_0) \geq \delta$, then $|\mathbf{x} \cdot \mathbf{u}| \gg \frac{1}{\psi(\|\mathbf{x}\|)\|\mathbf{x}\|^\gamma}$

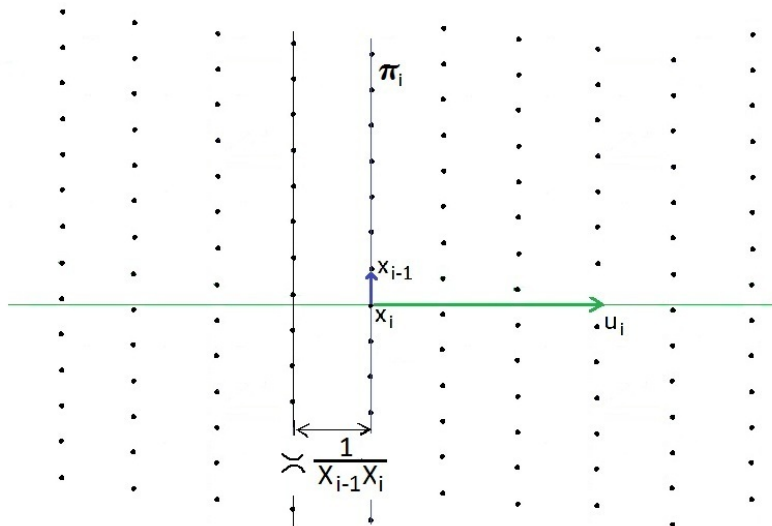
The lattice of integers points in π_i



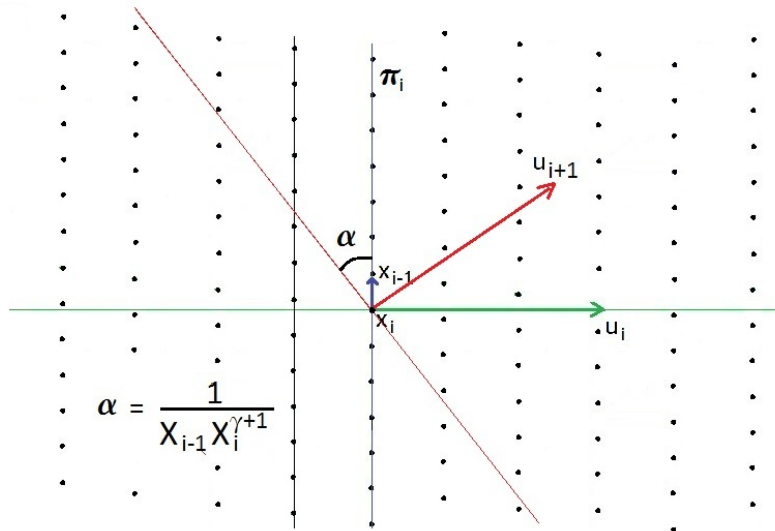
Side view



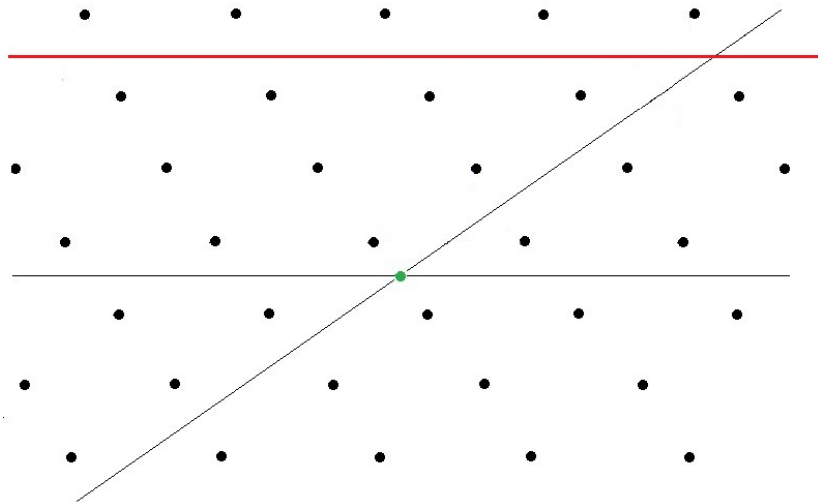
The closest parallel plane with integers points in it



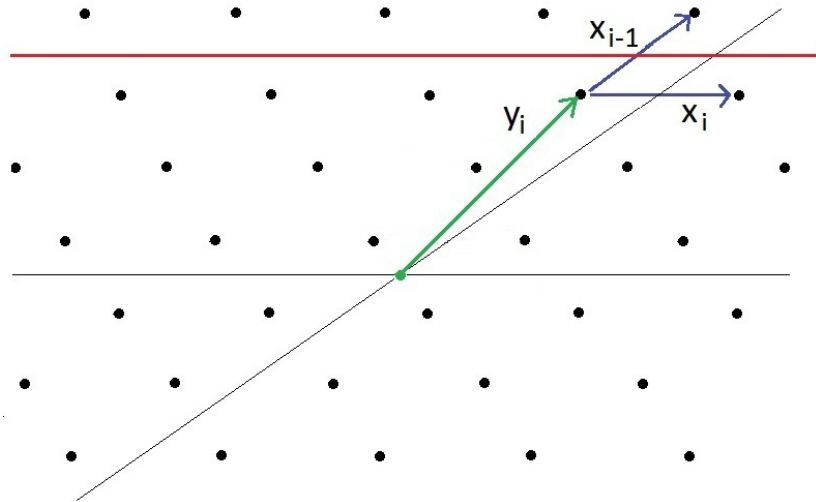
Rotating $\mathbf{u}_i$ by a small angle α about $\mathbb{R}\mathbf{x}_i$



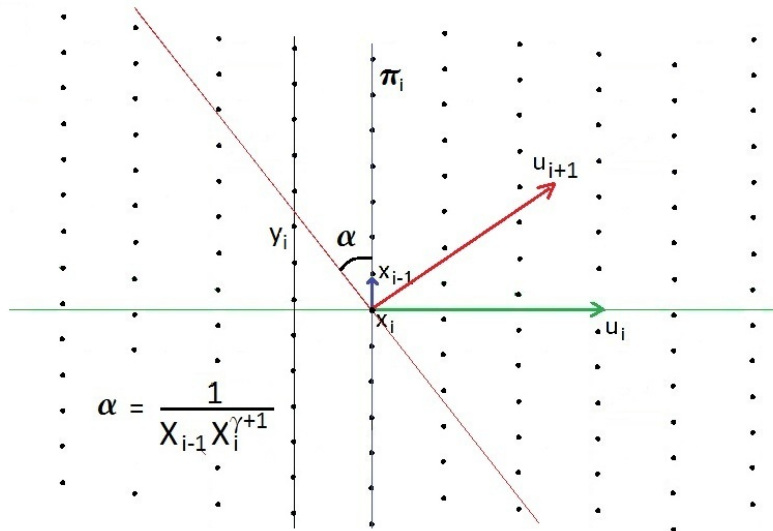
The integer points in the closest parallel plane



Choosing the point y_i



Back to the side view



Adjusting the angle to get $\mathbf{x}_{i+1}$

