

On Schmidt and Summerer parametric geometry of numbers

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Four classical exponents of approximation

Dirichlet (1842). Let $\underline{\xi} = (1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$. For each $Q \geq 1$,

- a) there exist $x_1, \dots, x_n \in \mathbb{Z}$ not all zero such that

$$|x_1| \leq Q \quad \text{and} \quad |x_1\xi_2 - x_2|, \dots, |x_1\xi_n - x_n| \leq Q^{-1/(n-1)},$$

- b) there exist $y_1, \dots, y_n \in \mathbb{Z}$ not all zero such that

$$|y_2|, \dots, |y_n| \leq Q \quad \text{and} \quad |y_1 + y_2\xi_2 + \dots + y_n\xi_n| \leq Q^{-(n-1)}.$$

- a) Define $\lambda(\underline{\xi}) = \sup$ of all $\lambda \geq 0$ such that

$$|x_1| \leq Q \quad \text{and} \quad |x_1\xi_2 - x_2|, \dots, |x_1\xi_n - x_n| \leq Q^{-\lambda}$$

has a nonzero solution $(x_1, \dots, x_n) \in \mathbb{Z}^n$ for arbitrarily large Q .

$\hat{\lambda}(\underline{\xi}) =$ same but for all sufficiently large values of Q .

- b) Define $\tau(\underline{\xi}) = \sup$ of all $\tau \geq 0$ such that

$$|y_2|, \dots, |y_n| \leq Q \quad \text{and} \quad |y_1 + y_2\xi_2 + \dots + y_n\xi_n| \leq Q^{-\tau}.$$

has a nonzero solution $(y_1, \dots, y_n) \in \mathbb{Z}^n$ for arbitrarily large Q .

$\hat{\tau}(\underline{\xi}) =$ same but for all sufficiently large values of Q .

Case of dimension $n = 3$

Let $1, \xi_2, \xi_3 \in \mathbb{R}$ be linearly independent over $\mathbb{Q}$. Put $\underline{\xi} = (1, \xi_2, \xi_3)$.

Dirichlet (1842): $\frac{1}{2} \leq \hat{\lambda}(\underline{\xi})$ and $2 \leq \hat{\tau}(\underline{\xi})$

Khintchine (1926–28): $\frac{\tau(\underline{\xi})}{\tau(\underline{\xi}) + 2} \leq \lambda(\underline{\xi}) \leq \frac{\tau(\underline{\xi}) - 1}{2}$

Jarník (1938): $\hat{\lambda}(\underline{\xi}) = 1 - \frac{1}{\hat{\tau}(\underline{\xi})}$

M. Laurent (2009): The set of all quadruples $(\lambda(\underline{\xi}), \hat{\lambda}(\underline{\xi}), \tau(\underline{\xi}), \hat{\tau}(\underline{\xi}))$ is

$$\left\{ (\lambda, \hat{\lambda}, \tau, \hat{\tau}); 2 \leq \hat{\tau} \leq \infty, \hat{\lambda} = 1 - \frac{1}{\hat{\tau}}, \frac{\tau(\hat{\tau} - 1)}{\tau + \hat{\tau}} \leq \lambda \leq \frac{\tau - \hat{\tau} + 1}{\hat{\tau}} \right\}$$

Geometry of numbers (Minkowski, 1889)

Let $\mathcal{C}$ be a (Minkowski) convex body in $\mathbb{R}^n$, i.e. a compact convex neighborhood $\mathcal{C}$ of 0, symmetric with respect to 0 (i.e. $-\mathcal{C} = \mathcal{C}$).

Theorem (Minkowski's first convex body theorem)

If $\text{vol}(\mathcal{C}) \geq 2^n$, then $\mathcal{C} \cap \mathbb{Z}^n \neq \emptyset$.

Example:

$$\mathcal{C} : \begin{cases} \max\{|y_2|, \dots, |y_n|\} \leq Q \\ |y_1 + y_2\xi_2 + \dots + y_n\xi_n| \leq Q^{-(n-1)} \end{cases} \quad \text{has volume } 2^n$$

So, it contains a non-zero integer point $(y_1, \dots, y_n)$.

Successive minima

Let $\mathcal{C}$ be a (Minkowski) convex body in $\mathbb{R}^n$.

For $j = 1, \dots, n$, the j -th minimum of $\mathcal{C}$, denoted $\lambda_j(\mathcal{C})$ is the smallest $\lambda > 0$ such that $\lambda\mathcal{C}$ contains at least j linearly independent elements of $\mathbb{Z}^n$.

- $\lambda_1(\mathcal{C}) \leq \dots \leq \lambda_n(\mathcal{C})$
- there exist $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{Z}^n$, linearly independent, such that

$$\mathbf{x}_j \in \lambda_j(\mathcal{C})\mathcal{C} \quad (1 \leq j \leq n).$$

Theorem (Minkowski's second convex body theorem)

$$\frac{2^n}{n!} \leq \lambda_1(\mathcal{C}) \cdots \lambda_n(\mathcal{C}) \text{vol}(\mathcal{C}) \leq 2^n.$$

Schmidt and Summerer theory (2009, 2013)

Let $\mathbf{u} \in \mathbb{R}^n$ with Euclidean norm $\|\mathbf{u}\| = 1$. Define

$$\mathcal{C}(Q) = \mathcal{C}_{\mathbf{u}}(Q) = \left\{ \mathbf{x} \in \mathbb{R}^n ; \|\mathbf{x}\| \leq 1, |\mathbf{x} \cdot \mathbf{u}| \leq \frac{1}{Q} \right\} \quad (Q \geq 1),$$

$$L_i(q) = \log \lambda_i(\mathcal{C}_{\mathbf{u}}(e^q)) \quad (q \geq 0, 1 \leq i \leq n)$$

Classical exponents of approximation can be expressed in terms of

$$\underline{\varphi}_i := \liminf_{q \rightarrow \infty} \frac{L_i(q)}{q} \quad \text{and} \quad \overline{\varphi}_i := \limsup_{q \rightarrow \infty} \frac{L_i(q)}{q} \quad i = 1, \dots, n.$$

Using their theory, they recover

- Jarník's and Khintchine's transference principles,
- their refinements by Bugeaud, Laurent and Schmidt,
- constraints by Laurent (2009) on the 4 main exponents when $n = 3$.

Basic properties of $L_1, \dots, L_n$

$$\mathcal{C}(Q) = \{\mathbf{x} \in \mathbb{R}^n; \|\mathbf{x}\| \leq 1, |\mathbf{x} \cdot \mathbf{u}| \leq Q^{-1}\}, \quad L_j(q) = \log \lambda_j(\mathcal{C}(e^q))$$

$$1) \quad \boxed{L_1(q) \leq \dots \leq L_n(q)}$$

$$\begin{aligned} \text{This yields a map } \mathbf{L}_{\mathbf{u}}: [0, \infty) &\longrightarrow \Delta_n \\ q &\longmapsto (L_1(q), \dots, L_n(q)) \end{aligned}$$

where $\Delta_n = \{(a_1, \dots, a_n) \in \mathbb{R}^n; a_1 \leq \dots \leq a_n\}$.

$$2) \quad \text{vol}(\mathcal{C}(Q)) \asymp Q^{-1} \Rightarrow \prod \lambda_i(\mathcal{C}(Q)) \asymp Q$$

$$\Rightarrow \boxed{\sum L_i(q) = q + \mathcal{O}(1)}$$

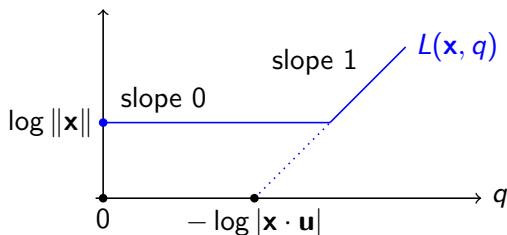
$$3) \quad \boxed{L_1, \dots, L_n \text{ are continuous and monotone increasing}}$$

The trajectory of a point

$$\mathcal{C}(Q) = \{\mathbf{x} \in \mathbb{R}^n; \|\mathbf{x}\| \leq 1, |\mathbf{x} \cdot \mathbf{u}| \leq Q^{-1}\}$$

For each $\mathbf{x} \in \mathbb{Z}^n \setminus \{0\}$, we define

- $\lambda(\mathbf{x}, Q) := \min \{\lambda > 0; \mathbf{x} \in \lambda \mathcal{C}(Q)\} = \max \{\|\mathbf{x}\|, Q|\mathbf{x} \cdot \mathbf{u}|\}$
- $L(\mathbf{x}, q) := \log \lambda(\mathbf{x}, e^q) = \max \{\log \|\mathbf{x}\|, q + \log |\mathbf{x} \cdot \mathbf{u}|\}$

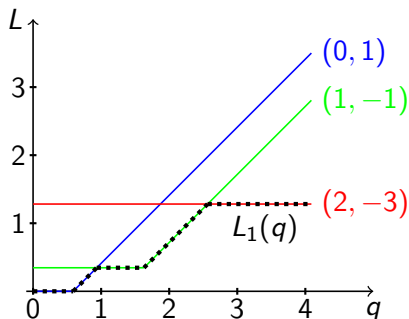


The first minimum

$$L(\mathbf{x}, q) = \max \{ \log \|\mathbf{x}\|, q + \log |\mathbf{x} \cdot \mathbf{u}| \}$$

4) $\Rightarrow L_1(q) = \min \{ L(\mathbf{x}, q); \mathbf{x} \in \mathbb{Z}^n \setminus \{0\} \} \quad (q \geq 0)$
is continuous and piecewise linear with slopes 0 and 1

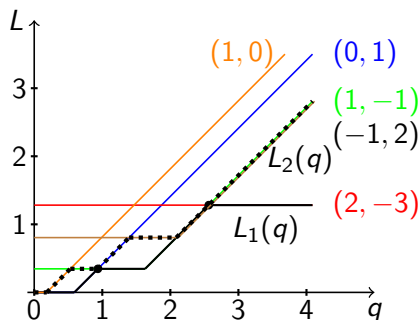
Example: $n = 2$, $\mathbf{u} = (3, 2)/\sqrt{13}$



Local maxima of L_1

- 5) $L_1(q) = L_2(q)$ at each q where L_1 changes slope from 1 to 0

Example: $n = 2$, $\mathbf{u} = (3, 2)/\sqrt{13}$



Compound convex bodies (Mahler, 1955)

- $\bigwedge^k \mathbb{R}^n$ can be made into an Euclidean space such that, for each orthonormal basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ of $\mathbb{R}^n$, the set

$$\{\mathbf{e}_{i_1} \wedge \dots \wedge \mathbf{e}_{i_k} ; 1 \leq i_1 < \dots < i_k \leq n\}$$

is an orthonormal basis of $\bigwedge^k \mathbb{R}^n$.

- Write $\mathbb{R}^n = U \perp \langle \mathbf{u} \rangle_{\mathbb{R}}$ where $U = \mathbf{u}^\perp$,

$$\bigwedge^k \mathbb{R}^n = \bigwedge^k U \perp W^{(k)} \quad \text{where} \quad W^{(k)} = \bigwedge^{k-1} U \wedge \langle \mathbf{u} \rangle_{\mathbb{R}},$$

$$\mathcal{C}^{(k)}(Q) = \left\{ \alpha \in \bigwedge^k \mathbb{R}^n ; \|\alpha\| \leq 1, \|\text{proj}_{W^{(k)}}(\alpha)\| \leq Q^{-1} \right\},$$

$$L_j^{(k)}(q) = \log \lambda_j(\mathcal{C}^{(k)}(e^q)) \quad \left(1 \leq j \leq \binom{n}{k} \right)$$

- Up to $\mathcal{O}(1)$, the minima $L_j^{(k)}(q)$ are a permutation of

$$L_{i_1}(q) + \dots + L_{i_k}(q) \quad (1 \leq i_1 < \dots < i_k \leq n).$$

Implications on the minima $L_1, \dots, L_n$

In particular, we have:

- $L_1^{(k)}(q) = L_1(q) + \dots + L_k(q) + \mathcal{O}(1)$
- $L_2^{(k)}(q) = L_1(q) + \dots + L_{k-1}(q) + L_{k+1}(q) + \mathcal{O}(1)$
- $\Rightarrow \boxed{L_2^{(k)}(q) - L_1^{(k)}(q) = L_{k+1}(q) - L_k(q) + \mathcal{O}(1)} \quad (0 < k < n)$

On the other hand, since

$$\mathcal{C}^{(k)}(Q) = \left\{ \alpha \in \wedge^k \mathbb{R}^n ; \|\alpha\| \leq 1, \|\text{proj}_{W^{(k)}}(\alpha)\| \leq Q^{-1} \right\}$$

the function

- $L_1^{(k)}$ is continuous and piecewise linear with slopes 0 and 1.
- At each point q where it changes slope from 1 to 0, we have

$$L_2^{(k)}(q) = L_1^{(k)}(q) \implies L_{k+1}(q) = L_k(q) + \mathcal{O}(1)$$

(n, γ) -systems

Definition. An (n, γ) -system is a map $\mathbf{P} = (P_1, \dots, P_n): [0, \infty) \rightarrow \mathbb{R}^n$ which satisfies the following conditions.

- (S1) $P_j(q) \leq P_{j+1}(q) + \gamma \quad (1 \leq j < n, 0 \leq q).$
- (S2) $P_j(q_1) \leq P_j(q_2) + \gamma \quad (1 \leq j \leq n, 0 \leq q_1 \leq q_2).$
- (S3) For $j = 1, \dots, n$, the function $M_j := P_1 + \dots + P_j: [0, \infty) \rightarrow \mathbb{R}$ is continuous and piecewise linear with slopes 0 and 1.
- (S4) $M_n(q) = q \quad (0 \leq q).$
- (S5) If, for $j \in \{1, \dots, n-1\}$, the function M_j changes slope from 1 to 0 at a point $q > 0$, then $P_{j+1}(q) \leq P_j(q) + \gamma.$

(S3) and (S4) $\Rightarrow P_1$ and P_n have slopes 0 and 1

$P_2, \dots, P_{n-1}$ have slopes $-1, 0$ and 1

The theorem of Schmidt and Summerer

Theorem (Schmidt-Summerer, 2013)

Let $\mathbf{u} \in \mathbb{R}^n$ with $\|\mathbf{u}\| = 1$. There exist $\gamma > 0$ and an (n, γ) -system $\mathbf{P}: [0, \infty) \rightarrow \mathbb{R}^n$ such that

$$\|\mathbf{P}(q) - \mathbf{L}_{\mathbf{u}}(q)\|_{\infty} \leq \gamma \quad \text{for each } q \geq 0.$$

Proof. Define

$$M_k(q) := \begin{cases} 0 & \text{if } k = 0, \\ L_1^{(k)}(q) & \text{if } 0 < k < n, \\ q & \text{if } k = n. \end{cases} \quad \text{and} \quad P_k(q) = M_k(q) - M_{k-1}(q).$$

Ideal case

Definition. An $(n, 0)$ -system is a map $\mathbf{P} = (P_1, \dots, P_n): [q_0, \infty) \rightarrow \mathbb{R}^n$ which satisfies the following conditions.

- (1) $P_1(q) \leq \dots \leq P_n(q) \quad (q_0 \leq q)$
- (2) $P_1, \dots, P_n$ are continuous and piecewise linear with slopes 0 and 1
- (3) $M_j := P_1 + \dots + P_j$ has slopes 0 and 1 ($1 \leq j \leq n$)
- (4) $M_n(q) = P_1(q) + \dots + P_n(q) = q \quad (q_0 \leq q).$
- (5) If, for $j \in \{1, \dots, n-1\}$, the function M_j changes slope from 1 to 0 at a point $q > q_0$, then $P_{j+1}(q) = P_j(q).$

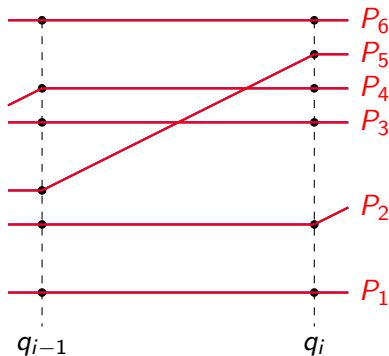
Let $\delta > 0$. We say that $\mathbf{P}$ is a *rigid system with mesh δ* if, for $q = q_0$ and for each $q > q_0$ at which at least one of the functions $M_1, \dots, M_{n-1}$ changes slope from 0 to 1, the coordinates of $\mathbf{P}(q)$ form a strictly increasing sequence of integer multiples of δ .

The combined graph of a rigid system

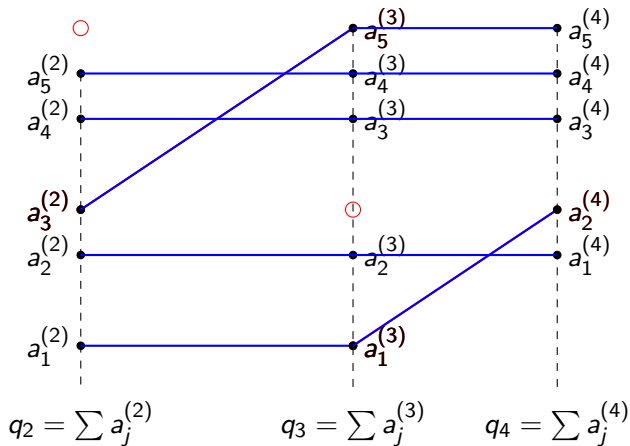
Let $\mathbf{P} = (P_1, \dots, P_n): [q_0, \infty) \rightarrow \mathbb{R}^n$ be a rigid system with mesh δ .

Denote by $q_1 < q_2 < \dots$ the points where at least one of the sums $M_j = P_1 + \dots + P_j$ changes slope from 0 to 1.

The combined graph of $P_1, \dots, P_n$ has the following shape:



Example ($n = 5$)



Canevas of mesh δ

A rigid system of mesh δ is equivalent to the following data:

- an element $s \in \{1, 2, 3, \dots\} \cup \{\infty\}$,
- for each index i with $0 \leq i < s$, a pair of integers k_i and ℓ_i with
$$1 \leq k_i < \ell_i \leq n \quad \text{and} \quad k_{i-1} \leq \ell_i \text{ if } i \geq 1,$$
- for each index i with $0 \leq i < s$, a point $(a_1^{(i)}, \dots, a_n^{(i)})$ whose coordinates form a strictly increasing sequence of multiples of δ with

$$(a_1^{(i-1)}, \dots, \widehat{a_{k_{i-1}}^{(i-1)}}, \dots, a_n^{(i-1)}) = (a_1^{(i)}, \dots, \widehat{a_{\ell_i}^{(i)}}, \dots, a_n^{(i)})$$

$$\text{and } a_{\ell_i}^{(i)} > a_{k_{i-1}}^{(i-1)}.$$

Just set

$$q_i = a_1^{(i)} + \dots + a_n^{(i)} \quad (0 \leq i < s)$$

and connect each $(q_{i-1}, a_j^{(i-1)})$ with the appropriate $(q_i, a_h^{(i)})$.

Approximation by rigid systems

A combinatorial argument shows:

Theorem A

Fix $\gamma, \delta > 0$, and let $\tilde{\mathbf{P}}: [0, \infty) \rightarrow \mathbb{R}^n$ be an (n, γ) -system. There exists a rigid system $\mathbf{P}: [q_0, \infty) \rightarrow \mathbb{R}^n$ such that

$$\|\tilde{\mathbf{P}}(q) - \mathbf{P}(q)\|_{\infty} \leq c_1 \quad \text{for each } q \geq q_0,$$

where $c_1 = c_1(\gamma, \delta, n)$.

In view of the theorem of Schmidt and Summerer, this gives.

Corollary

Let $\mathbf{u} \in \mathbb{R}^n$ with $\|\mathbf{u}\| = 1$. For each $\delta > 0$ there exists a rigid system $\mathbf{P}: [q_0, \infty) \rightarrow \mathbb{R}^n$ with mesh δ such that

$$\|\mathbf{P}(q) - \mathbf{L}_{\mathbf{u}}(q)\|_{\infty} \leq c_2 \quad \text{for each } q \geq q_0,$$

where $c_2 = c_2(\delta, n)$.

A first observation towards the inverse problem

Let $\mathbf{P}: [q_0, \infty) \rightarrow \mathbb{R}^n$ be a rigid system with mesh $\boxed{\delta \geq 5n\epsilon}$ where $\boxed{\epsilon = 2^n \log(n)}$. Consider its associated sequences

$$(\mathbf{a}^{(i)})_{0 \leq i < s}, \quad (k_i)_{0 \leq i < s}, \quad (\ell_i)_{0 \leq i < s} \quad \text{and} \quad (q_i)_{0 \leq i < s}.$$

Theorem B

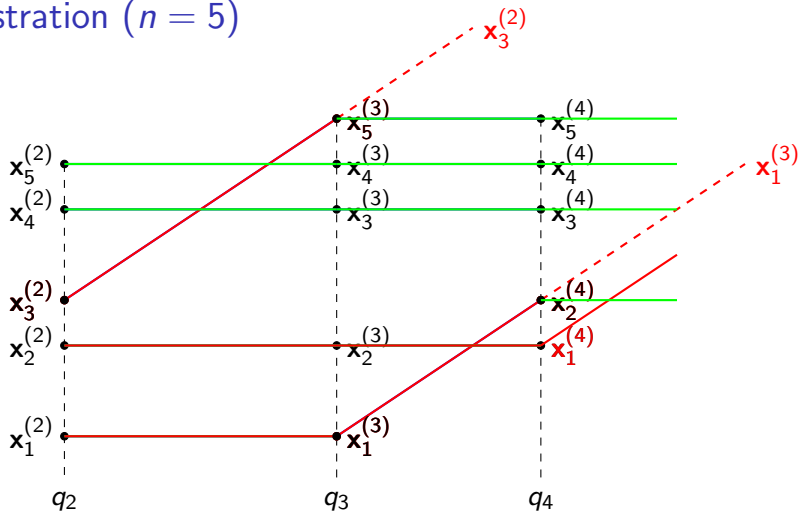
Suppose that there exists a unit vector $\mathbf{u} \in \mathbb{R}^n$ such that

$$\sup_{q \geq q_0} \|\mathbf{P}(q) - \mathbf{L}_{\mathbf{u}}(q)\|_{\infty} \leq \epsilon.$$

Then, for each i with $0 \leq i < s$, there exist $\mathbf{x}_1^{(i)}, \dots, \mathbf{x}_n^{(i)} \in \mathbb{Z}^n$ such that

- $0 \leq \log |\det(\mathbf{x}_1^{(i)}, \dots, \mathbf{x}_n^{(i)})| \leq 2n\epsilon,$
- $(\mathbf{x}_1^{(i-1)}, \dots, \widehat{\mathbf{x}_{k_{i-1}}^{(i-1)}}, \dots, \mathbf{x}_n^{(i-1)}) = (\mathbf{x}_1^{(i)}, \dots, \widehat{\mathbf{x}_{\ell_i}^{(i)}}, \dots, \mathbf{x}_n^{(i)})$ if $i \geq 1,$
- *above each $q \in [q_i, q_{i+1})$, the trajectories of $\mathbf{x}_1^{(i)}, \dots, \mathbf{x}_n^{(i)}$ are at distance at most ϵ from $\mathbf{P}(q)$.*

Illustration ($n = 5$)



$$(x_1^{(2)}, x_2^{(2)}, \widehat{x_3^{(2)}}, x_4^{(2)}, x_5^{(2)}) = (x_1^{(3)}, x_2^{(3)}, x_3^{(3)}, x_4^{(3)}, \widehat{x_5^{(3)}})$$

$$(\widehat{x_1^{(3)}}, x_2^{(3)}, x_3^{(3)}, x_4^{(3)}, x_5^{(3)}) = (x_1^{(4)}, \widehat{x_2^{(4)}}, x_3^{(4)}, x_4^{(4)}, x_5^{(4)})$$

Inverse problem

Theorem C

For each sufficiently large $\delta > 0$ and each rigid system $P: [q_0, \infty) \rightarrow \mathbb{R}_+^n$ with mesh δ , there exists a unit vector $\mathbf{u} \in \mathbb{R}^n$ such that

$$\|\mathbf{P}(q) - \mathbf{L}_{\mathbf{u}}(q)\|_{\infty} \leq c_3 \quad \text{for each } q \geq q_0,$$

where $c_3 = c_3(n) > 0$.

Proof: We need construct appropriate n -tuples $(\mathbf{x}_1^{(i)}, \dots, \mathbf{x}_n^{(i)})$ for the given canvas but we don't know $\mathbf{u} \dots$

We proceed recursively:

- We choose each $(\mathbf{x}_1^{(i)}, \dots, \mathbf{x}_n^{(i)})$ to be a basis of $\mathbb{Z}^n$,
- making $\mathbf{x}_1^{(i)}, \dots, \widehat{\mathbf{x}_{k_i}^{(i)}}, \dots, \mathbf{x}_n^{(i)}$ “almost orthogonal” of the right norms.

We take $\mathbf{u}$ to be the limit of the unit vectors perpendicular to the spaces spanned by the latter $(n - 1)$ -tuples.

Application to classical exponents of approximation

Corollary

Let $n \in \mathbb{N}$ with $n \geq 2$. We have a bijection between the set of quadruples

$$(\tau(\underline{\xi}), \hat{\tau}(\underline{\xi}), \hat{\lambda}(\underline{\xi}), \lambda(\underline{\xi}))$$

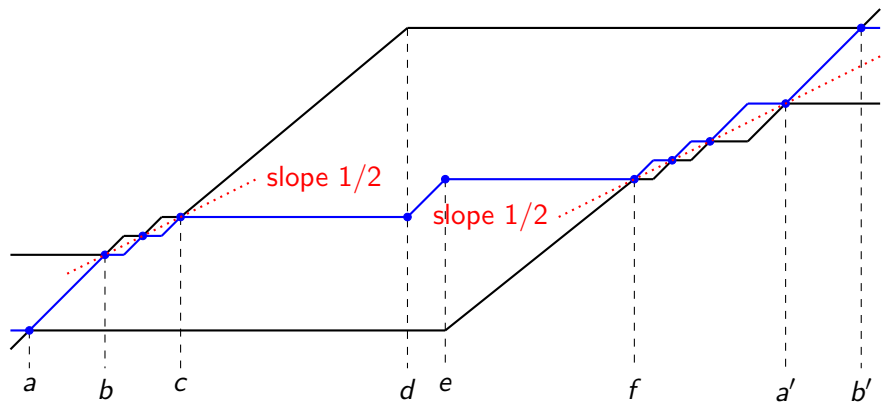
where $\underline{\xi}$ runs through all points $\underline{\xi} = (1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$ with $\mathbb{Q}$ -linearly independent coordinates and the set of quadruples

$$\left(\liminf_{q \rightarrow \infty} \frac{P_1(q)}{q}, \limsup_{q \rightarrow \infty} \frac{P_1(q)}{q}, \liminf_{q \rightarrow \infty} \frac{P_n(q)}{q}, \limsup_{q \rightarrow \infty} \frac{P_n(q)}{q} \right)$$

where $\mathbf{P} = (P_1, \dots, P_n)$ runs through all rigid systems with mesh 1 and $\lim_{q \rightarrow \infty} P_1(q) = \infty$. This map is given by

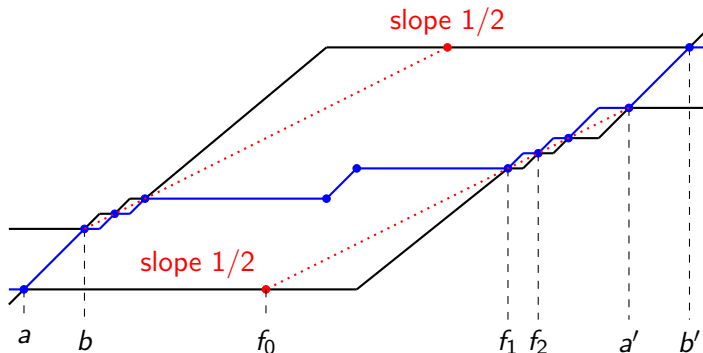
$$(\tau, \hat{\tau}, \hat{\lambda}, \lambda) \mapsto \left(\frac{1}{\tau + 1}, \frac{1}{\hat{\tau} + 1}, \frac{\hat{\lambda}}{\hat{\lambda} + 1}, \frac{\lambda}{\lambda + 1} \right).$$

Basic pattern for a rigid system in dimension $n = 3$



The knowledge of P_2 determines the whole map $\mathbf{P} : [q_0, \infty) \rightarrow \mathbb{R}^3$.

Example: Jarník's transference principle

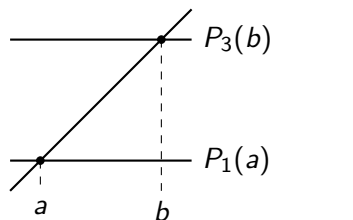


We have $0 \leq P_1(q)/q \leq 1$. So the ratio $P_1(q)/q$

- decreases when the slope of P_1 is 0 and increases when it is 1,
- has a local max. at $f_1, f_2, \dots, a'$ where P_1 changes slope from 1 to 0
- achieves its maximum on $[a, a']$ at a or at a' .

Similarly $P_3(q)/q$ achieves its minimum on $[b, b']$ at b or at b' .

Jarník's transference principle (continued)


$$\Rightarrow \begin{cases} a = 2P_1(a) + P_3(b) \\ b = P_1(a) + 2P_3(b) \\ b - a = P_3(b) - P_1(a) \end{cases}$$

$$\Rightarrow (P_1(a), P_3(b)) = \left(\frac{2a - b}{3}, \frac{2b - a}{3} \right)$$

$$\Rightarrow \left(2 - 3 \frac{P_1(a)}{a} \right) \left(2 - 3 \frac{P_3(b)}{b} \right) = 1$$

$$\Rightarrow \left(2 - 3 \limsup \frac{P_1(a)}{a} \right) \left(2 - 3 \liminf \frac{P_3(b)}{b} \right) = 1$$

$$\Rightarrow \left(2 - \frac{3}{\hat{\tau} + 1} \right) \left(2 - \frac{3\hat{\lambda}}{\hat{\lambda} + 1} \right) = 1 \quad \Rightarrow \quad \boxed{\hat{\lambda} = 1 - \frac{1}{\hat{\tau}}}.$$

Thank you!