

SMALL VALUE ESTIMATES FOR THE ADDITIVE GROUP

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*À Michel Waldschmidt,
infatigable voyageur et grand communicateur,
avec mes meilleurs vœux et ma plus grande estime,
à l'occasion de son soixantième anniversaire.*

We generalize Gel'fond's criterion for algebraic independence to the context of a sequence of polynomials whose first derivatives take small values on large subsets of a fixed subgroup of $\mathbb{C}$, instead of just one point (one extension deals with a subgroup of $\mathbb{C}^\times$).

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1. Introduction

A typical proof of algebraic independence starts with the construction of a sequence of auxiliary polynomials taking small values at many points of a finitely generated subgroup Γ of a commutative algebraic group G defined over some algebraic extension of $\mathbb{Q}$. This data is analyzed by applying sequentially a criterion of algebraic independence and a zero estimate. The criterion of algebraic independence first looks at each value individually while the zero estimate is used to ensure that the polynomials do not vanish on nearby points from a slight perturbation $\tilde{\Gamma}$ of Γ . The outcome is a lower bound for the transcendence degree over $\mathbb{Q}$ of the field K generated by the coordinates of the points of Γ .

For further progress, it would be desirable to have a tool that encompasses both the criterion and the zero estimate by looking at these small values globally as values of polynomials on the group G instead of looking at them one at a time, as

elements of the field K . In [10], we conjecture such a “small value estimate” for the group $\mathbb{G}_a \times \mathbb{G}_m$, and prove that it is equivalent to Schanuel’s conjecture. In [11], we further extend these ideas to the group $\mathbb{G}_a \times E$ where E is an elliptic curve defined over $\mathbb{Q}$.

The present paper mainly deals with small value estimates for the additive group $\mathbb{G}_a$ as a first step towards these conjectures. The following theorem provides an overview of our main results. In its formulation, the symbols i and j are restricted to integers. We also write $H(P)$ to denote the *height* of a polynomial $P \in \mathbb{Z}[T]$, and $P^{[j]}$ to denote its j th divided derivative (see Sec. 2 for the precise definitions).

Theorem 1.1. *Let ξ be a transcendental complex number, let β, σ, τ and ν be non-negative real numbers, let n_0 be a positive integer, and let $(P_n)_{n \geq n_0}$ be a sequence of non-zero polynomials in $\mathbb{Z}[T]$ satisfying $\deg(P_n) \leq n$ and $H(P_n) \leq \exp(n^\beta)$ for each $n \geq n_0$. The following six statements hold.*

- (1) *Let r be a non-zero rational number. Suppose that $\beta > 1$, $\sigma + \tau < 1$ and $\nu > 1 + \beta - \sigma - \tau$. Then for infinitely many n , we have*

$$\max\{|P_n^{[j]}(\xi + ir)|; 0 \leq i \leq n^\sigma, 0 \leq j \leq n^\tau\} > \exp(-n^\nu).$$

- (2) *Let r be a positive rational number with $r \neq 1$. Suppose that $\beta > 1 + \sigma$, $\sigma + \tau < 1$ and $\nu > 1 + \beta - \sigma - \tau$. Then for infinitely many n , we have*

$$\max\{|P_n^{[j]}(r^i \xi)|; 0 \leq i \leq n^\sigma, 0 \leq j \leq n^\tau\} > \exp(-n^\nu).$$

- (3) *Suppose that $\beta > 1$, $(3/4)\sigma + \tau < 1$ and $\nu > 1 + \beta - (3/4)\sigma - \tau$. Then for infinitely many n , we have*

$$\max\{|P_n^{[j]}(i\xi)|; 0 \leq i \leq n^\sigma, 0 \leq j \leq n^\tau\} > \exp(-n^\nu).$$

- (4) *Let r be a non-zero rational number. Suppose that $\beta > 1$, $(4/3)\sigma + \tau < 1$ and $\nu > 1 + \beta - (4/3)\sigma - \tau$. Then for infinitely many n , we have*

$$\max\{|P_n^{[j]}(i_1 \xi + i_2 r)|; 0 \leq i_1, i_2 \leq n^\sigma, 0 \leq j \leq n^\tau\} > \exp(-n^\nu).$$

- (5) *Let $\eta \in \mathbb{C}$ be algebraic over $\mathbb{Q}(\xi)$ with $\eta \notin \mathbb{Q}\xi$. Suppose that $\beta > 1$, $(3/2)\sigma + \tau < 1$ and $\nu > 1 + \beta - \sigma - \tau$. Then for infinitely many n , we have*

$$\max\{|P_n^{[j]}(i_1 \xi + i_2 \eta)|; 0 \leq i_1, i_2 \leq n^\sigma, 0 \leq j \leq n^\tau\} > \exp(-n^\nu).$$

- (6) *Let $\eta \in \mathbb{C}$. Suppose that $\beta > 1$, $\sigma < 1$ and $\nu > 3 + \beta - (11/4)\sigma$. Then for infinitely many n , we have*

$$\max\{|P_n(i\xi + \eta)|; 0 \leq i \leq n^\sigma\} > \exp(-n^\nu).$$

To reach the same conclusions using a standard version of Gel’fond’s criterion (see [1, 13]), one requires in each case a lower bound on ν of the form $\nu > 1 + \beta$. Using instead [7, Proposition 1], which represents a version of Gel’fond’s criterion with multiplicities, one needs $\nu > 1 + \beta - \tau$. Thus, for $\sigma > 0$, the statements (1)–(5) improve on the known versions of Gel’fond’s criterion in their respective contexts.

For $\sigma = 0$ however, they essentially reduce to [7, Proposition 1]. The statements (1) and (2) are listed first because, together with (4), they are the simplest to prove and also because, in view of Dirichlet box principle, they show a best possible dependence in the parameter ν (see Proposition A.1 from Appendix A). They can be viewed as extensions of [12, Theorem 2.3] dealing with polynomials of unbounded degree. However, the main goal of the paper is to prove (3) and (5) whose forms are closest to the conjectural small value estimate of [10]. In Sec. 8, we provide an extension of (5) involving linear combinations of m points from a field of transcendence one over $\mathbb{Q}$, instead of just two points. By contrast, (6) is a first step towards working with elements from a field of higher transcendence degree. Coming back to (3), Dirichlet box principle shows that it would be false for a value of ν smaller than $1 + \beta - \sigma - \tau$. This shows a gap of $\sigma/4$ compared to our actual lower bound on ν . We have not been able to reduce it further. Similarly (4), (5) and (6) show, respectively, a gap of $(2/3)\sigma$, σ and $2 - (7/4)\sigma$ in the dependence in ν compared to the box principle (see Appendix A).

The proof of all results proceeds by contradiction and ultimately rely on a version of Gel'fond's criterion for algebraic independence that we recall in the next section. Section 3, inspired from [7, Sec. 6], deals with estimates for the resultant of polynomials in one variable taking into account the absolute values of the first derivatives of these polynomials at the points of a finite set E . Section 4 borrow ideas from the proof of zero estimates to provide upper bounds for the degree and height of an irreducible polynomial dividing the first derivatives of polynomials of the form $P(aT + b)$ where $P \in \mathbb{Q}[T]$ is fixed and (a, b) runs through a finite subset of $\mathbb{Q}_+ \times \mathbb{Q}$. These tools are combined in Sec. 5 to prove (1), (2) and (4). Statement (5) is proved in Sec. 8 in a more general form involving subgroups of arbitrary rank. Besides the tools that have already been mentioned, its proof also uses the following result established in Sec. 7 as a consequence of a combinatorial result from Sec. 6:

Theorem 1.2. *Let β , δ and μ be positive real numbers with $\mu < 1 < \beta$, let A be the set of all prime numbers p with $p \leq n^\mu$, let P be a non-zero polynomial of $\mathbb{Q}[T]$ of degree at most n and height at most $\exp(n^\beta)$ with $P(0) \neq 0$, and let Q be the greatest common divisor of the polynomials $P(aT)$ with $a \in A$. If n is sufficiently large as a function of β , δ and μ , we have $\deg(Q) \leq n^{1-\mu+\delta}$ and $H(Q) \leq \exp(n^{\beta-\mu+\delta})$.*

The proof of (3) given in Sec. 10 furthermore uses the following result proved as a consequence of another combinatorial statement from Sec. 9, related to Zarankiewicz problem:

Theorem 1.3. *Let α , β , δ and μ be positive real numbers with $2\mu < \alpha < \beta$. For each integer $n \geq 1$, let A_n denote the set of all prime numbers p with $p \leq n^\mu$, and B_n denote the set of all prime numbers p with $n^\mu < p \leq n^{2\mu}$. For infinitely many n , there exists no non-zero polynomial $P \in \mathbb{Z}[T]$ of degree at most n^α and height at most $\exp(n^\beta)$ satisfying $\prod_{a \in A_n} \prod_{b \in B_n} |P(ab\xi)| \leq \exp(-n^{\alpha+\beta+2\mu+\delta})$.*

Finally, (6) is proved in Sec. 11 as a consequence of (3) after elimination of η through a resultant, upon observing that this resultant as well as its first derivatives are small at multiples of ξ .

Sketch of proof of (3). In order to help the reader find the way through this paper, we conclude this Introduction by a brief sketch of proof of (3). We proceed by contradiction, assuming on the contrary that for each sufficiently large n the polynomial P_n satisfies $|P_n^{[j]}(i\xi)| \leq \exp(-n^\nu)$ for $i = 1, \dots, [n^\sigma]$ and $j = 0, 1, \dots, [n^\tau]$. Without loss of generality, after division of each P_n by a suitable power of T , we may assume that these polynomials do not vanish at 0. Define A_n and B_n as in Theorem 1.3 for the choice of $\mu = \sigma/4$, and let Q_n be the greatest common divisor of the polynomials $P_n^{[j]}(aT)$ with $a \in A_n$ and $j = 0, 1, \dots, [n^{7/2}]$. Upon observing that the latter family of polynomials take small values at the points $ab\xi$ with $a \in A_n$ and $b \in B_n$, along with their derivatives of order at most $[n^{7/2}]$, we deduce that $\prod_{a \in A_n} \prod_{b \in B_n} |Q_n(ab\xi)| \leq \exp(-n^{1+\beta-\tau+5\delta})$ for some positive δ which is independent of n . By Theorem 1.2, we further know that Q_n has degree at most $n^{1-\sigma/4+\delta}$ and height at most $\exp(n^{\beta-\sigma/4+\delta})$. By a standard linearization process described in Sec. 2, we deduce that Q_n admits an irreducible factor R_n satisfying $\prod_{a \in A_n} \prod_{b \in B_n} |R_n(ab\xi)| \leq \exp(-n^{\sigma/4-\tau+3\delta}(n^\beta \deg(R_n) + n \log H(R_n)))$. By independent means, we also find that R_n has degree at most $n^{1-\sigma/4-\tau+\delta}$ and height at most $\exp(n^{\beta-\sigma/4-\tau+\delta})$. Then, we deduce that there exists a power S_n of R_n whose degree and height satisfy the same estimates, with moreover $\prod_{a \in A_n} \prod_{b \in B_n} |S_n(ab\xi)| \leq \exp(-n^{1+\beta-2\tau+3\delta})$. This contradicts Theorem 1.3.

2. Notation and Preliminaries

We denote respectively by $\mathbb{Q}^\times$ and $\mathbb{C}^\times$ the multiplicative groups of $\mathbb{Q}$ and $\mathbb{C}$. We also write $\mathbb{Q}_+$ for the multiplicative group of positive rational numbers, and $\mathbb{N}^*$ for the set of positive integers. We denote by $|E|$ the cardinality of a set E . Given subsets A and B of $\mathbb{C}$, we write $A + B$ (respectively, AB) to denote the set of all sums $a + b$ (respectively, products ab) with $a \in A$ and $b \in B$. Throughout this paper, the symbols i, j, k are restricted to integers. For any field L , any $P \in L[T]$ and any $j \geq 0$, we denote by $P^{[j]}(T)$ the coefficient of U^j in the expansion of $P(T + U)$ as a polynomial in $L[T][U]$. When L has characteristic zero, which is the case of interest here, this is simply the quotient by $j!$ of the j th derivative of P .

We define the *norm* $\|\mathbf{x}\|$ of any point $\mathbf{x}$ in $\mathbb{C}^n$ to be its maximum norm. Similarly, we define the *norm* $\|P\|$ of a polynomial $P \in \mathbb{C}[T_1, \dots, T_m]$ to be the maximum of the absolute values of its coefficients. When $\mathbf{x}$ is a non-zero element of $\mathbb{Q}^n$, we define its *content* $\text{cont}(\mathbf{x})$ to be the unique positive rational number r such that $r^{-1}\mathbf{x}$ is a *primitive* point of $\mathbb{Z}^n$, namely, a point of $\mathbb{Z}^n$ with relatively prime coordinates. We also define its *height* $H(\mathbf{x})$ to be the ratio $\|\mathbf{x}\|/\text{cont}(\mathbf{x})$. By extension, we define, respectively, the *content* $\text{cont}(P)$ and *height* $H(P)$ of a non-zero polynomial $P \in \mathbb{Q}[T_1, \dots, T_m]$ to be the content and height of its coefficient vector.

Accordingly, we have $H(P) = \|P\|/\text{cont}(P)$. This notion of height is *projective* as we have $H(a\mathbf{x}) = H(\mathbf{x})$ and $H(aP) = H(P)$ for any $a \in \mathbb{Q}^\times$. For a single rational number x , we adopt a slightly different convention, and define its *height* $H(x)$ to be the inhomogeneous height of x , namely, the height of the point $(1, x) \in \mathbb{Q}^2$. This gives $H(x) = \max(|p|, |q|)$ if p/q is the reduced form of x .

We will frequently use the well-known fact that for any $P_1, \dots, P_s \in \mathbb{C}[T]$ with product $P = P_1 \cdots P_s$, we have

$$e^{-\deg(P)} \|P\| \leq \|P_1\| \cdots \|P_s\| \leq e^{\deg(P)} \|P\|$$

[4, Chap. III, Sec. 4, Lemma 2]. As the content is a multiplicative function on $\mathbb{Q}[T] \setminus \{0\}$, it follows that, for non-zero polynomials $P_1, \dots, P_s \in \mathbb{Q}[T]$, the same inequalities hold with the norm replaced by the height. This means that the height is essentially multiplicative. In the sequel, we will also require the following lemma which formalizes a standard procedure of “linearization”:

Lemma 2.1. *Let c, n, ρ and X be positive real numbers with $e^n \leq X$, and let $\xi_1, \dots, \xi_s$ be a finite sequence of complex numbers, not necessarily distinct. Suppose that there exists a non-zero polynomial $P \in \mathbb{Q}[T]$ of degree at most ρn and height at most X^ρ satisfying*

$$\prod_{i=1}^s \frac{|P(\xi_i)|}{\text{cont}(P)} < (X^{\deg(P)} H(P)^n)^{-c/\rho} \quad (1)$$

or the stronger condition

$$\prod_{i=1}^s \frac{|P(\xi_i)|}{\text{cont}(P)} < X^{-2cn}. \quad (2)$$

Then,

(a) *there exists an irreducible factor R of P in $\mathbb{Q}[T]$ satisfying*

$$\prod_{i=1}^s \frac{|R(\xi_i)|}{\text{cont}(R)} < (X^{\deg(R)} H(R)^n)^{-c/(2\rho)}, \quad (3)$$

(b) *there exists an integer $k \geq 1$ such that the polynomial $Q = R^k$ satisfies*

$$\deg(Q) \leq \rho n, \quad H(Q) \leq X^{2\rho} \quad \text{and} \quad \prod_{i=1}^s \frac{|Q(\xi_i)|}{\text{cont}(Q)} < X^{-cn/4}. \quad (4)$$

Usually the data takes the form (2). In replacing it by the weaker condition (1), one gains that the right-hand side becomes essentially a multiplicative function of P . Part (b) of the lemma shows that not much is lost in the process, regardless of the value of ρ . However, for given c, n and X , the conclusion of Part (a) gets stronger for small values of ρ .

Proof. Upon replacing n by n/ρ , X by $X^{1/\rho}$ and c by $\rho^2 c$, we may assume without loss of generality that $\rho = 1$. We also note that the strict inequality in (1) implies that P is a non-constant polynomial.

- (a) Factor P as a product $P = R_1 \cdots R_u$ of irreducible polynomials of $\mathbb{Q}[T]$. Since $H(R_1) \cdots H(R_u) \leq e^{\deg(P)} H(P)$, the condition (1) implies

$$\prod_{j=1}^u \prod_{i=1}^s \frac{|R_j(\xi_i)|}{\text{cont}(R_j)} < (X^{\deg(P)} (e^{\deg(P)} H(P))^n)^{-c/2} \leq \prod_{j=1}^u (X^{\deg(R_j)} H(R_j)^n)^{-c/2}.$$

Therefore there is at least one index j for which the polynomial $R = R_j$ satisfies (3).

- (b) Since R divides P , we have $\deg(R) \leq n$ and $H(R) \leq e^n X \leq X^2$. Let $k \geq 1$ be the largest integer for which the polynomial $Q = R^k$ satisfies $\deg(Q) \leq n$ and $H(Q) \leq X^2$. Taking the k th power on both sides of (3), we obtain

$$\prod_{i=1}^u \frac{|Q(\xi_i)|}{\text{cont}(Q)} < (X^{\deg(Q)} H(R)^{kn})^{-c/2}.$$

If $\deg(Q) \geq n/2$, the right-hand side of this inequality is bounded above by $X^{-cn/4}$, and the conditions of (4) are all satisfied. Assume now that $\deg(Q) \leq n/2$. Then we have $\deg(R^{2k}) \leq n$, and the choice of k implies $H(R^{2k}) \geq X^2$. Since $H(R^{2k}) \leq e^n H(R)^{2k}$, we deduce that $H(R)^k \geq X^{1/2}$, and we reach the same conclusion. $\square$

We conclude this section by stating the version of Gel'fond's criterion for algebraic independence on which all our results ultimately rely.

Lemma 2.2. *Let α , β and δ be positive real numbers with $\beta \geq \alpha$, and let $\xi_1, \dots, \xi_m$ be a finite sequence of complex numbers which generate a field of transcendence degree 1 over $\mathbb{Q}$. For infinitely many integers n , there exists no polynomial $P \in \mathbb{Z}[T_1, \dots, T_m]$ of degree at most n^α and height at most $\exp(n^\beta)$ satisfying*

$$0 < |P(\xi_1, \dots, \xi_m)| \leq \exp(-n^{\alpha+\beta+\delta}).$$

This follows for example from [9, Theorem 2.11] or [8, Sec. 7, Corollary 3]. Alternatively, a standard norm argument reduces the proof of this result to the case $m = 1$ which is a direct consequence of [1, Theorem 1]. The fact that one can separate the estimates for the degree and height of the polynomials is an original observation of Brownawell and Waldschmidt which played a crucial role in their solution of Schneider's eighth problem [2, 13]. Note that, in the case $m = 1$, the condition $0 < |P(\xi_1)|$ can simply be replaced by $P \neq 0$ since ξ_1 is assumed to be transcendental over $\mathbb{Q}$.

3. Estimates for the Resultant

For any finite subset E of $\mathbb{C}$ with at least two points, we define

$$\delta_E = \min_{\xi' \neq \xi} |\xi' - \xi| \quad \text{and} \quad \Delta_E = \prod_{\xi' \neq \xi} |\xi' - \xi|^{1/2} \quad (5)$$

where both the minimum and the product are taken over all ordered pairs (ξ', ξ) of distinct elements of E . When E consists of one point, we put $\delta_E = \Delta_E = 1$. In the sequel, we will often use the crude estimate $\Delta_E \geq \min(1, \delta_E)^{(1/2)|E|^2}$. The main result of this section is the following.

Proposition 3.1. *Let $n, s, t \in \mathbb{N}^*$ with $n \geq st$, let E be a set of s complex numbers, let F and G be non-zero polynomials of $\mathbb{Q}[T]$ of degree at most n and let $Q \in \mathbb{Q}[T]$ be their greatest common divisor. For any pair of integers f and g with $\deg(F/Q) \leq f \leq n$ and $\deg(G/Q) \leq g \leq n$, we have*

$$H(Q)^{f+g} \prod_{\xi \in E} \left(\frac{|Q(\xi)|}{\|Q\|} \right)^t \leq c_1 H(F)^g H(G)^f \prod_{\xi \in E} \max_{0 \leq j < t} \left\{ \frac{|F^{[j]}(\xi)|}{\|F\|}, \frac{|G^{[j]}(\xi)|}{\|G\|} \right\}^t, \quad (6)$$

with $c_1 = e^{7n^2} (2 + c_E)^{4nst} \Delta_E^{-t^2}$, where $c_E = \max_{\xi \in E} |\xi|$ and Δ_E is defined above.

When $s = 1$, this is essentially [7, Lemma 13]. In other words, we can view the above proposition as an extension of the latter result dealing with values of polynomials and their derivatives at several points instead of one. The proof is similar in that it proceeds through estimations of the resultant of F/Q and G/Q . It will require several intermediate lemmas. Before going into this, we note the following corollary.

Corollary 3.2. *Let $n, s, t \in \mathbb{N}^*$ with $n \geq st$, let E be a set of s complex numbers, let $P_1, \dots, P_r \in \mathbb{Q}[T]$ be a finite sequence of $r \geq 2$ non-zero polynomials of degree at most n , and let $Q \in \mathbb{Q}[T]$ be their greatest common divisor. Then we have*

$$\prod_{\xi \in E} \left(\frac{|Q(\xi)|}{\text{cont}(Q)} \right)^t \leq e^{3n^2} c_1 \left(\max_{1 \leq i \leq r} H(P_i) \right)^{2n} \prod_{\xi \in E} \left(\max_{\substack{1 \leq i \leq r \\ 0 \leq j < t}} \frac{|P_i^{[j]}(\xi)|}{\text{cont}(P_i)} \right)^t, \quad (7)$$

where c_1 is as in Proposition 3.1.

Proof. Without loss of generality, we may assume that $P_1, \dots, P_r$ and Q have content 1, or equivalently that they are primitive polynomials of $\mathbb{Z}[T]$. We may also assume that $Q(\xi) \neq 0$ for each $\xi \in E$, and that Q is not the gcd of any proper subset of $\{P_1, \dots, P_r\}$. The latter condition implies that $r \leq n + 1$. According to [7, Lemma 12] there exist integers $a_1, \dots, a_r$ with $0 \leq a_i \leq n$ for $i = 1, \dots, r$ such that Q is the gcd of $F := P_1$ and $G := \sum_{i=1}^r a_i P_i$. Assuming, as we may, that

$a_1 = 0$, we find

$$\max\{H(F), H(G)\} \leq \max\left\{\|P_1\|, n \sum_{i=2}^r \|P_i\|\right\} \leq n^2 \max_{1 \leq i \leq r} H(P_i)$$

and similarly, for any $\xi \in E$ and any $j = 0, \dots, t-1$,

$$\max\{|F^{[j]}(\xi)|, |G^{[j]}(\xi)|\} \leq n^2 \max_{1 \leq i \leq r} |P_i^{[j]}(\xi)|.$$

Applying Proposition 3.1 with $f = g = n$, we then find

$$\begin{aligned} \prod_{\xi \in E} |Q(\xi)|^t &\leq H(Q)^{2n} \prod_{\xi \in E} \left(\frac{|Q(\xi)|}{\|Q\|} \right)^t \\ &\leq c_1 \max\{H(F), H(G)\}^{2n} \prod_{\xi \in E} \max_{0 \leq j < t} \max\{|F^{[j]}(\xi)|, |G^{[j]}(\xi)|\}^t \\ &\leq c_1 (n^2)^{2n+st} \left(\max_{1 \leq i \leq r} H(P_i) \right)^{2n} \prod_{\xi \in E} \left(\max_{\substack{1 \leq i \leq r \\ 0 \leq j < t}} |P_i^{[j]}(\xi)| \right)^t. \end{aligned}$$

The conclusion follows using $n^2 \leq e^n$ and $st \leq n$. $\square$

In order to prove our main Proposition 3.1, we start by establishing a simple technical lemma.

Lemma 3.3. *Let $n, t \in \mathbb{N}^*$, let $z, \xi \in \mathbb{C}$, and let $F \in \mathbb{C}[T]$ be a non-zero polynomial with $\deg(F) \leq n$. Then, for each integer $\ell \geq 0$, the polynomial $\tilde{F}(T) = (T-z)^\ell F(T)$ satisfies*

$$\max_{0 \leq j < t} \frac{|\tilde{F}^{[j]}(\xi)|}{\|\tilde{F}\|} \leq e^{\deg(\tilde{F})} (2 + |\xi|)^\ell \max_{0 \leq j < t} \frac{|F^{[j]}(\xi)|}{\|F\|}. \quad (8)$$

When $z = 0$, we can omit the factor $e^{\deg(\tilde{F})}$ in the upper bound.

Proof. For any $j \geq 0$, we have $\tilde{F}^{[j]}(\xi) = \sum_{h=0}^{\min(j, \ell)} \binom{\ell}{h} (\xi - z)^{\ell-h} F^{[j-h]}(\xi)$ and so,

$$\max_{0 \leq j < t} |\tilde{F}^{[j]}(\xi)| \leq (1 + |\xi - z|)^\ell \max_{0 \leq j < t} |F^{[j]}(\xi)|.$$

This leads to the required upper bound (8) since $\|\tilde{F}\| \geq e^{-\deg(\tilde{F})} \max\{1, |z|\}^\ell \|F\|$. When $z = 0$, we simply have $\|\tilde{F}\| = \|F\|$ and we may omit the factor $e^{\deg(\tilde{F})}$. $\square$

The next result is purely algebraic.

Lemma 3.4. *Let $m, s, t \in \mathbb{N}^*$ with $m \geq st$, let L be any field, let $Q \in L[T]$, and let $\xi_1, \dots, \xi_s$ be s distinct elements of L . Denote by $L[T]_{\leq m-1}$ the vector space of polynomials of $L[T]$ of degree at most $m-1$, and let φ and ψ denote the L -linear*

maps from $L[T]_{\leq m-1}$ to L^m which send a polynomial $P \in L[T]_{\leq m-1}$ to the points $\varphi(P)$ and $\psi(P)$ of L^m whose k th coordinates are, respectively, given by

$$\varphi(P)_k = \begin{cases} (QP)^{[j]}(\xi_i) & \text{if } k = i + js \text{ with } 1 \leq i \leq s \text{ and } 0 \leq j < t, \\ P^{[k-1]}(0) & \text{if } st < k \leq m, \end{cases}$$

$$\psi(P)_k = P^{[k-1]}(0) \quad \text{for } k = 1, \dots, m.$$

Then, for any choice of polynomials $P_1, \dots, P_m \in L[T]_{\leq m-1}$, we have

$$\det(\varphi(P_1), \dots, \varphi(P_m)) = \pm \Delta^{t^2} \left(\prod_{i=1}^s Q(\xi_i) \right)^t \det(\psi(P_1), \dots, \psi(P_m)) \quad (9)$$

where $\Delta = \prod_{1 \leq i < j \leq s} (\xi_j - \xi_i)$ if $s \geq 2$, and $\Delta = 1$ if $s = 1$.

Proof. It suffices to show that (9) holds for at least one choice of L -linearly independent polynomials $P_1, \dots, P_m$. Put $E(T) = (T - \xi_1) \cdots (T - \xi_s)$ and, for each $k = 1, \dots, m$, define

$$P_k = \begin{cases} E(T)^j (T - \xi_1) \cdots (T - \xi_{i-1}) & \text{if } k = i + js \text{ with } 1 \leq i \leq s \text{ and } 0 \leq j < t, \\ T^{k-1} & \text{if } st < k \leq m. \end{cases}$$

Then, each P_k is a monic polynomial of degree $k - 1$ and so the $m \times m$ matrix whose rows are $\psi(P_1), \dots, \psi(P_m)$ is lower triangular with all its diagonal entries equal to 1. This gives $\det(\psi(P_1), \dots, \psi(P_m)) = 1$. We claim that the matrix with rows $\varphi(P_1), \dots, \varphi(P_m)$ has a block decomposition of the form $\begin{pmatrix} U & 0 \\ M & I \end{pmatrix}$ where U is an upper triangular $st \times st$ matrix and I denotes the identity matrix of size $(m - st) \times (m - st)$. To prove this, we fix indices k, k' with $1 \leq k, k' \leq m$. If $k, k' > st$, we find $\varphi(P_k)_{k'} = 0$ when $k' \neq k$ and $\varphi(P_k)_{k'} = 1$ when $k' = k$. If $k' > st \geq k$, we also find $\varphi(P_k)_{k'} = 0$ since P_k has degree $k - 1$. Suppose now that $k' < k \leq st$. We can write $k = i + js$ and $k' = i' + j's$ with $1 \leq i, i' \leq s$ and $0 \leq j, j' < t$. Since $k' < k$, either we have $j' = j$ and $i' < i$ or we have $j' < j$. In both cases, we find that $(T - \xi_{i'})^{j'+1}$ divides QP_k and so $\varphi(P_k)_{k'} = 0$. We also note that

$$\varphi(P_k)_k = Q(\xi_i) E'(\xi_i)^j (\xi_i - \xi_1) \cdots (\xi_i - \xi_{i-1}).$$

This proves the claim and also provides the value of the diagonal elements of the matrix U . Consequently we have $\det(\varphi(P_1), \dots, \varphi(P_m)) = \det(U)$ where

$$\det(U) = \prod_{j=0}^{t-1} \prod_{i=1}^s Q(\xi_i) E'(\xi_i)^j (\xi_i - \xi_1) \cdots (\xi_i - \xi_{i-1}) = \pm \Delta^{t^2} \prod_{i=1}^s Q(\xi_i)^t.$$

Thus (9) holds for the present choice of $P_1, \dots, P_m$ and therefore it holds in general. $\square$

Lemma 3.5. Let $n, s, t \in \mathbb{N}^*$, and let E be a set of s complex numbers. Let $F, G \in \mathbb{C}[T]$ be non-zero polynomials of degree at most n , and let $Q \in \mathbb{C}[T]$ be their greatest

common divisor. Put $A = F/Q$, $B = G/Q$, $a = \deg(A)$, $b = \deg(B)$ and $m = a + b$. Finally, assume that $m \geq st$. Then we have

$$|\text{Res}(A, B)| \prod_{\xi \in E} \left(\frac{|Q(\xi)|}{\|Q\|} \right)^t \leq c_2 \|A\|^b \|B\|^a \prod_{\xi \in E} \max_{0 \leq j < t} \left\{ \frac{|F^{[j]}(\xi)|}{\|F\|}, \frac{|G^{[j]}(\xi)|}{\|G\|} \right\}^t \quad (10)$$

with $c_2 = m!(e(2 + c_E))^{nst} \Delta_E^{-t^2}$ and $c_E = \max_{\xi \in E} |\xi|$.

Proof. Let $\xi_1, \dots, \xi_s$ denote the s elements of E . By definition of the resultant, we have $\text{Res}(A, B) = \det(\psi(P_1), \dots, \psi(P_m))$ where ψ is defined as in Lemma 3.4 for the choice of $L = \mathbb{C}$, and where $P_1, \dots, P_m$ stand for the sequence of polynomials

$$A(T), TA(T), \dots, T^{b-1}A(T), B(T), TB(T), \dots, T^{a-1}B(T).$$

Applying Lemma 3.4, we deduce that

$$|\text{Res}(A, B)| \prod_{i=1}^s |Q(\xi_i)|^t = \Delta_E^{-t^2} |\det M|$$

where M is the $m \times m$ matrix with rows $\varphi(P_1), \dots, \varphi(P_m)$ for the map $\varphi: \mathbb{C}[T]_{\leq m-1} \rightarrow \mathbb{C}^m$ defined in the lemma. Let $\tilde{M}$ be the matrix obtained from M by dividing each of its first b rows by $\|A\|$ and each of its last a rows by $\|B\|$. Then, except in its first st columns, all coefficients of $\tilde{M}$ have absolute value at most 1. This implies

$$|\det M| = \|A\|^b \|B\|^a |\det \tilde{M}| \leq m! \|A\|^b \|B\|^a C_1 \cdots C_{st},$$

where, for each $k = 1, \dots, st$, we denote by C_k the maximum norm of the k th column of $\tilde{M}$. Fix a choice of k as above and write it in the form $k = i + js$ with $1 \leq i \leq s$ and $0 \leq j < t$. Applying Lemma 3.3 with $z = 0$ together with the estimate $\|F\| \leq e^n \|A\| \|Q\|$, we find that the absolute values of the first b elements in the k th column of $\tilde{M}$ are bounded above by:

$$\begin{aligned} \max_{0 \leq \ell < b} \frac{|(T^\ell F)^{[j]}(\xi_i)|}{\|A\|} &\leq e^n \|Q\| \max_{0 \leq \ell < b} \frac{|(T^\ell F)^{[j]}(\xi_i)|}{\|F\|} \\ &\leq (e(2 + c_E))^n \|Q\| \max_{0 \leq j < t} \frac{|F^{[j]}(\xi_i)|}{\|F\|}. \end{aligned}$$

Upon replacing b by a , A by B and F by G in the above inequalities, we also get an upper bound for the absolute values of the last a elements in the k th column of $\tilde{M}$. This gives

$$C_k \leq (e(2 + c_E))^n \|Q\| \max_{0 \leq j < t} \left\{ \frac{|F^{[j]}(\xi_i)|}{\|F\|}, \frac{|G^{[j]}(\xi_i)|}{\|G\|} \right\}.$$

The conclusion follows. $\square$

Lemma 3.6. *Lemma 3.5 still holds if the hypothesis $m \geq st$ is replaced by $n \geq st$ and the constant c_2 in (10) is replaced by $c_3 = (2n)!(e(2 + c_E))^{4nst} \Delta_E^{-t^2}$ with the same value for c_E .*

Proof. Since $c_3 \geq c_2$, we may assume without loss of generality that $m < st \leq n$. Cauchy's inequalities show that all coefficients of a polynomial have their absolute value bounded above by the supremum norm of the polynomial on the unit circle of the complex plane. Applying this to the polynomial B , we deduce that there exists $z \in \mathbb{C}$ with $|z| = 1$ such that $\|B\| \leq |B(z)|$. Put

$$\tilde{F}(T) = (T - z)^{st-a-b} F(T) \quad \text{and} \quad \tilde{A}(T) = \frac{\tilde{F}(T)}{Q(T)} = (T - z)^{st-a-b} A(T).$$

As $B(z) \neq 0$, we still have $\gcd(\tilde{F}, G) = Q$. Since $\deg(\tilde{A}B) = st$ and since $\tilde{F}$ and G both have degree at most $2n$, Lemma 3.5 gives

$$|\text{Res}(\tilde{A}, B)| \prod_{\xi \in E} \left(\frac{|Q(\xi)|}{\|Q\|} \right)^t \leq c \|\tilde{A}\|^b \|B\|^{st-b} \prod_{\xi \in E} \max_{0 \leq j < t} \left\{ \frac{|\tilde{F}^{[j]}(\xi)|}{\|\tilde{F}\|}, \frac{|G^{[j]}(\xi)|}{\|G\|} \right\}^t$$

where $c = (st)!(e(2 + c_E))^{2nst} \Delta_E^{-t^2}$. On the other hand, using the fact that the resultant is multiplicative in each of its arguments and that $\text{Res}(T - z, B) = \pm B(z)$, we find

$$|\text{Res}(\tilde{A}, B)| = |B(z)|^{st-a-b} |\text{Res}(A, B)| \geq \|B\|^{st-a-b} |\text{Res}(A, B)|.$$

The conclusion follows by combining the above two inequalities together with $\|\tilde{A}\| \leq 2^{st} \|A\|$ and the estimate

$$\max_{0 \leq j < t} \frac{|\tilde{F}^{[j]}(\xi)|}{\|\tilde{F}\|} \leq (e(2 + c_E))^{2n-b} \max_{0 \leq j < t} \frac{|F^{[j]}(\xi)|}{\|F\|},$$

valid for each $\xi \in E$, which follows from Lemma 3.3 using $\deg(\tilde{F}) \leq n + st - b \leq 2n - b$. $\square$

Proof of Proposition 3.1. Since both sides of the inequality (6) stay invariant under multiplication of F , G and Q by non-zero rational numbers, we may assume without loss of generality that F , G and Q are primitive polynomials of $\mathbb{Z}[T]$. Put $A = F/Q$ and $B = G/Q$. Then, A and B are relatively prime primitive polynomials of $\mathbb{Z}[T]$. In particular, their resultant $\text{Res}(A, B)$ is a non-zero integer, and so we have $|\text{Res}(A, B)| \geq 1$. Since $\|A\| = H(A)$ and $\|B\| = H(B)$ are also positive integers, we deduce from Lemma 3.6 that

$$\prod_{\xi \in E} \left(\frac{|Q(\xi)|}{\|Q\|} \right)^t \leq c_3 H(A)^f H(B)^g \prod_{\xi \in E} \max_{0 \leq j < t} \left\{ \frac{|F^{[j]}(\xi)|}{\|F\|}, \frac{|G^{[j]}(\xi)|}{\|G\|} \right\}^t$$

for any pair of integers f and g satisfying the conditions of the proposition. We also note that $c_3 \leq e^{5n^2} (2 + c_E)^{4nst} \Delta_E^{-t^2}$ since $(2n)! \leq e^{n^2}$ and $st \leq n$. The conclusion then follows from the fact that $H(A) \leq e^n H(F)/H(Q)$ and $H(B) \leq e^n H(G)/H(Q)$ since F and G both have degree at most n . $\square$

4. Estimates for Translates of Polynomials

For each $a \in \mathbb{Q}_+$ and each $b \in \mathbb{Q}$, we denote by $\lambda_{a,b}$ the automorphism of $\mathbb{Q}[T]$ which maps a polynomial $P \in \mathbb{Q}[T]$ to

$$(\lambda_{a,b}P)(T) = P(aT + b).$$

This provides an injective map from $\mathbb{Q}_+ \times \mathbb{Q}$ to the group of automorphisms of $\mathbb{Q}[T]$, whose image is a subgroup $\mathcal{L}$ of that group. We define the *height* of an element $\lambda_{a,b}$ of $\mathcal{L}$ by

$$H(\lambda_{a,b}) = H(1, a, b).$$

Since $\lambda_{a,b}^{-1} = \lambda_{1/a, -b/a}$ and $H(1, 1/a, -b/a) = H(a, 1, -b) = H(1, a, b)$, we have $H(\lambda^{-1}) = H(\lambda)$ for any $\lambda \in \mathcal{L}$. A similar computation shows that $H(\lambda\lambda') \leq 2H(\lambda)H(\lambda')$ for any $\lambda, \lambda' \in \mathcal{L}$. Finally, we let the group $\mathcal{L}$ act on $\mathbb{C}$ by

$$\lambda_{a,b} \cdot \xi = a\xi + b,$$

so that $(\lambda P)(\xi) = P(\lambda \cdot \xi)$ for any $\lambda \in \mathcal{L}$ and any $\xi \in \mathbb{C}$.

Lemma 4.1. *Let $\lambda \in \mathcal{L}$, let P be a non-zero polynomial of $\mathbb{Q}[T]$, and let n be an upper bound for the degree of P . Then, we have $\deg(\lambda P) = \deg(P) \leq n$,*

$$(3H(\lambda))^{-n} \leq \frac{H(\lambda P)}{H(P)} \leq (3H(\lambda))^n \quad \text{and} \quad H(\lambda)^{-n} \leq \frac{\text{cont}(\lambda P)}{\text{cont}(P)} \leq H(\lambda)^n.$$

Proof. Without loss of generality, we may assume that P is a primitive polynomial of $\mathbb{Z}[T]$ of degree n . It is clear that $\deg \lambda P = n$. Choose $a \in \mathbb{Q}_+$ and $b \in \mathbb{Q}$ such that $\lambda = \lambda_{a,b}$, and let q be the least common denominator of a and b , so that $H(\lambda) = |q| \max\{1, |a|, |b|\}$. Since $q^n \lambda(P) = q^n P(aT + b)$ has integer coefficients, we find

$$H(\lambda P) \leq |q|^n \|P(aT + b)\| \leq |q|^n (1 + |a| + |b|)^n \|P\| \leq (3H(\lambda))^n H(P),$$

and $\text{cont}(\lambda P) \geq |q|^{-n} \geq H(\lambda)^{-n} = H(\lambda)^{-n} \text{cont}(P)$. The remaining inequalities follow from the above with λ replaced by λ^{-1} and P replaced by λP , using $H(\lambda^{-1}) = H(\lambda)$. $\square$

Lemma 4.2. *Let $a \in \mathbb{Q}_+$, $b \in \mathbb{Q}$, and let R be an irreducible polynomial of $\mathbb{Q}[T]$. Then, $\lambda_{a,b}R$ is also an irreducible polynomial of $\mathbb{Q}[T]$. Moreover, it is an associate of R if and only if either we have $a \neq 1$ and R is a rational multiple of $(a-1)T + b$, or we have $(a, b) = (1, 0)$.*

Proof. The first assertion follows from the fact that $\lambda_{a,b}$ is an automorphism of $\mathbb{Q}[T]$ and that any automorphism of an integral domain maps units to units and irreducible elements to irreducible elements.

Suppose that $\lambda_{a,b}R$ is an associate of R . Then $\lambda_{a,b}$ permutes the roots of R in $\mathbb{C}$, and so there is an integer $k \geq 1$ for which $\lambda_{a,b}^k$ fixes all roots of R . This means

that the roots of R are also roots of the polynomial $\lambda_{a,b}^k(T) - T$. If $a = 1$, this polynomial is the constant kb and so we must have $b = 0$. If $a \neq 1$, it is a non-zero polynomial of $\mathbb{Q}[T]$ of degree 1 with leading coefficient $a^k - 1 \neq 0$ (since $a > 0$). In this case, R must also be a polynomial of degree 1 and $\lambda_{a,b}$ fixes its root. This root must therefore be $b/(1 - a)$ and so R is a rational multiple of $(a - 1)T + b$. The converse is clear. $\square$

Lemma 4.3. *Let $n, s, t \in \mathbb{N}^*$ with $st \leq n$, let $\mathcal{A}$ be a finite subset of $\mathcal{L}$ of cardinality at least s , let P be a non-zero polynomial of $\mathbb{Q}[T]$ of degree at most n , and let R be an irreducible polynomial of $\mathbb{Q}[T]$. Suppose that R divides $\lambda(P^{[i]})$ for each $\lambda \in \mathcal{A}$ and each $i = 0, 1, \dots, t - 1$. Then, we have*

$$\deg(R) \leq n/(st) \quad \text{and} \quad H(R) \leq \left(3 \max_{\lambda \in \mathcal{A}} H(\lambda)\right)^{2n/(st)} H(P)^{1/(st)}. \quad (11)$$

Proof. By Lemma 4.2, the polynomials $\lambda^{-1}R$ with $\lambda \in \mathcal{A}$ are all irreducible. Suppose first that no two of them are associates. Let $\lambda \in \mathcal{A}$. Since R divides $\lambda(P^{[i]})$ for $i = 0, 1, \dots, t - 1$, we deduce that $\lambda^{-1}R$ divides $P^{[i]}$ for the same values of i , and therefore that $(\lambda^{-1}R)^t$ divides P . This being true for each $\lambda \in \mathcal{A}$, we conclude that $\prod_{\lambda \in \mathcal{A}} (\lambda^{-1}R)^t$ divides P . Since, by Lemma 4.1, the polynomials $\lambda^{-1}R$ have the same degree as R and height at least $(3H(\lambda))^{-\deg(R)} H(R)$, we first deduce that $\deg(R) \leq n/(st)$ and then that

$$H(P) \geq e^{-n} \prod_{\lambda \in \mathcal{A}} H(\lambda^{-1}R)^t \geq \left(3e \max_{\lambda \in \mathcal{A}} H(\lambda)\right)^{-n} H(R)^{st},$$

which is stronger than (11).

Suppose now that there exist two distinct elements λ' and λ'' of $\mathcal{A}$ for which $\lambda'R$ and $\lambda''R$ are associates. Then, R is an associate of λR for the composite $\lambda = (\lambda')^{-1}\lambda''$. Since λ is not the identity, Lemma 4.2 shows that R has degree 1, and using the explicit description of R given by this lemma we find $H(R) \leq 2H(\lambda) \leq 4H(\lambda')H(\lambda'')$. Then, the inequalities (11) are again satisfied because of the hypothesis $n \geq st$. $\square$

5. Basic Small Value Estimates

In the preceding section, we introduced a group $\mathcal{L}$ of automorphisms of $\mathbb{Q}[T]$ and an action of $\mathcal{L}$ on $\mathbb{C}$. With this notation, we now prove the following proposition which constitutes the first step in the proof of each of the first five statements of Theorem 1.1.

Proposition 5.1. *Let $n, t \in \mathbb{N}^*$, let $\mathcal{A}$ be a non-empty finite subset of $\mathcal{L}$, and let E be a non-empty finite subset of $\mathbb{C}$. Suppose that $|E|t \leq n$. Moreover, let P be a non-zero polynomial of $\mathbb{Z}[T]$ of degree at most n , and let Q denote the greatest common divisor in $\mathbb{Q}[T]$ of the polynomials $\lambda(P^{[i]})$ with $\lambda \in \mathcal{A}$ and $0 \leq i < t$. Then,*

upon putting

$$\delta_P = \max\{|P^{[j]}(\lambda \cdot \xi)|; \lambda \in \mathcal{A}, \xi \in E, 0 \leq j < 2t\},$$

we have

$$\prod_{\xi \in E} \left(\frac{|Q(\xi)|}{\text{cont}(Q)} \right)^t \leq (e^4 c_{\mathcal{A}} (2 + c_E))^{4n^2} \Delta_E^{-t^2} H(P)^{2n} \delta_P^{|E|t},$$

where $c_{\mathcal{A}} = \max_{\lambda \in \mathcal{A}} H(\lambda)$ and $c_E = \max_{\xi \in E} |\xi|$.

Proof. We apply Corollary 3.2 to the family of polynomials $\lambda(P^{[i]})$ with $\lambda \in \mathcal{A}$ and $0 \leq i < t$. Each of them has degree at most n and, using Lemma 4.1, we find that their height and content satisfy

$$\begin{aligned} H(\lambda(P^{[i]})) &\leq (3H(\lambda))^n H(P^{[i]}) \leq (6H(\lambda))^n H(P) \leq (6c_{\mathcal{A}})^n H(P), \\ \text{cont}(\lambda(P^{[i]})) &\geq H(\lambda)^{-n} \text{cont}(P^{[i]}) \geq H(\lambda)^{-n} \text{cont}(P) \geq c_{\mathcal{A}}^{-n}, \end{aligned}$$

where the last step in the second estimate comes from the hypothesis that P has integer coefficients. Moreover, upon writing $\lambda = \lambda_{a,b}$ with $a \in \mathbb{Q}_+$ and $b \in \mathbb{Q}$, we find, for each $\xi \in E$ and $j = 0, 1, \dots, t-1$,

$$(\lambda(P^{[i]}))^{[j]}(\xi) = \binom{i+j}{i} a^j P^{[i+j]}(a\xi + b) = \binom{i+j}{i} a^j P^{[i+j]}(\lambda \cdot \xi).$$

Since $|a| \leq H(\lambda) \leq c_{\mathcal{A}}$, we deduce that

$$\frac{|(\lambda(P^{[i]}))^{[j]}(\xi)|}{\text{cont}(\lambda(P^{[i]}))} \leq 2^{2t} c_{\mathcal{A}}^{t+n} \delta_P \leq (2c_{\mathcal{A}})^{2n} \delta_P.$$

According to Corollary 3.2, this implies that

$$\prod_{\xi \in E} \left(\frac{|Q(\xi)|}{\text{cont}(Q)} \right)^t \leq c((6c_{\mathcal{A}})^n H(P))^{2n} ((2c_{\mathcal{A}})^{2n} \delta_P)^{|E|t},$$

where $c = e^{10n^2} (2 + c_E)^{4n|E|t} \Delta_E^{-t^2}$. The conclusion follows using $|E|t \leq n$. $\square$

The next proposition analyzes the outcome of the preceding result through the linearization process of Sec. 2, with the help of the degree and height estimates of Lemma 4.3.

Proposition 5.2. *Let $n, s, t \in \mathbb{N}^*$ with $st \leq n$, let $\mathcal{A}$ be a finite subset of $\mathcal{L}$, and let E be a finite subset of $\mathbb{C}$. Suppose that $\min(|\mathcal{A}|, |E|) \geq s$. Moreover, let X be a real number with*

$$X \geq \max\{3^n, c_{\mathcal{A}}^n, (2 + c_E)^n, \delta_E^{-s^2 t^2 / n}\}, \quad (12)$$

where $c_{\mathcal{A}}$ and c_E are as in Proposition 5.1, and assume that there exists a non-zero polynomial P of $\mathbb{Z}[T]$ of degree at most n and height at most X satisfying

$$\max\{|P^{[j]}(\lambda \cdot \xi)|; \lambda \in \mathcal{A}, \xi \in E, 0 \leq j < 2t\} \leq X^{-\kappa n / (st)} \quad (13)$$

for some real number $\kappa > 27$. Then, there exist a primary polynomial $S \in \mathbb{Q}[T]$ and a point $\xi \in E$ with

$$\deg(S) \leq \frac{5n}{st}, \quad H(S) \leq X^{10/(st)} \quad \text{and} \quad \frac{|S(\xi)|}{\text{cont}(S)} \leq X^{-\kappa'n/(st)^2}, \quad (14)$$

where $\kappa' = (\kappa - 27)/16$.

Proof. Upon replacing E by a smaller subset if necessary, we may assume without loss of generality that $|E| = s$. Let Q be a greatest common divisor in $\mathbb{Q}[T]$ of the polynomials $\lambda(P^{[i]})$ with $\lambda \in \mathcal{A}$ and $0 \leq i < t$. Since $\Delta_E \geq \min(1, \delta_E)^{s^2}$, the condition (12) implies in particular that $\Delta_E^{-t^2} \leq \min(1, \delta_E)^{-s^2 t^2} \leq X^n$, and so Proposition 5.1 gives

$$\prod_{\xi \in E} \left(\frac{|Q(\xi)|}{\text{cont}(Q)} \right)^t \leq X^{(27-\kappa)n} = X^{-16\kappa'n}. \quad (15)$$

Choose any $\lambda \in \mathcal{A}$. Since Q divides λP , we find

$$\deg(Q) \leq \deg(\lambda P) \leq n \quad \text{and} \quad H(Q) \leq e^n H(\lambda P) \leq (3ec_{\mathcal{A}})^n H(P) \leq X^4, \quad (16)$$

where the estimates for the degree and height of λP come from Lemma 4.1. Therefore Lemma 2.1 applies to the present situation with $\rho = 4$ and $c = 8\kappa'$. It produces an irreducible factor R of Q in $\mathbb{Q}[T]$ with

$$\prod_{\xi \in E} \left(\frac{|R(\xi)|}{\text{cont}(R)} \right)^t \leq (X^{\deg(R)} H(R)^n)^{-\kappa'}. \quad (17)$$

Since $|E| = s$, there also exists a point $\xi \in E$ for which

$$\frac{|R(\xi)|}{\text{cont}(R)} \leq (X^{\deg(R)} H(R)^n)^{-\kappa'/(st)}. \quad (18)$$

Moreover, according to Lemma 4.3, the polynomial R satisfies

$$\deg(R) \leq \frac{n}{st} \quad \text{and} \quad H(R) \leq (3c_{\mathcal{A}})^{2n/(st)} H(P)^{1/(st)} \leq X^{5/(st)},$$

like all irreducible factors of Q . Applying Lemma 2.1 to R with $\rho = 5/(st)$ and $c = 5\kappa'/(st)^2$, we deduce that some power S of R has the required properties (14). $\square$

It is not possible in general to improve significantly on the estimates (16). We will see however that this can be done when the set $\mathcal{A}$ contains a collection of automorphisms of the form $\lambda_{a,0}$ with a in a multiplicatively independent subset of $\mathbb{Q}_+$ (see Sec. 7). This then brings a significant improvement on (17) which automatically carries to (18). The later step going from (17) to (18) can also be improved in some instances by noting that the values of R on the set E cannot be uniformly small (see Sec. 10).

The next result proves the statements (1) and (4) of Theorem 1.1 by choosing $\sigma_1 = 0$ and $\sigma_2 = \sigma$ for Part (1), and $\sigma_1 = \sigma_2 = \sigma$ for Part (4).

Theorem 5.3. *Let ξ be a transcendental complex number, let r be a non-zero rational number, and let $\beta, \sigma_1, \sigma_2, \tau, \nu$ be non-negative real numbers with*

$$\beta > 1, \quad \sigma_1 \leq 3\sigma_2, \quad (1/3)\sigma_1 + \sigma_2 + \tau < 1 \quad \text{and} \quad \nu > 1 + \beta - (1/3)\sigma_1 - \sigma_2 - \tau.$$

Then, there are arbitrarily large values of n for which there exists no non-zero polynomial $P \in \mathbb{Z}[T]$ of degree at most n and height at most $\exp(n^\beta)$ with

$$\max\{|P^{[j]}(i_1\xi + i_2r)|; 1 \leq i_1 \leq n^{\sigma_1}, 1 \leq i_2 \leq n^{\sigma_2}, 0 \leq j \leq n^\tau\} \leq \exp(-n^\nu).$$

Proof. Suppose on the contrary that such a polynomial P exists for each sufficiently large integer n , and choose a real number $\delta > 0$ such that

$$\nu \geq 1 + \beta - (1/3)\sigma_1 - \sigma_2 - \tau + 4\delta \quad \text{and} \quad 1 \geq (1/3)\sigma_1 + \sigma_2 + \tau + 3\delta.$$

For a fixed large integer n and a corresponding polynomial P , define

$$\begin{aligned} \mathcal{A} &= \{\lambda_{i_1, i_2r}; 1 \leq i_1 \leq n^{\sigma_1/3}, 0 \leq i_2 \leq (1/2)n^{\sigma_2}\}, \\ E &= \{i_1\xi + i_2r; 1 \leq i_1 \leq n^{2\sigma_1/3}, 0 \leq i_2 \leq (1/2)n^{\sigma_2 - \sigma_1/3}\}, \end{aligned}$$

where the values of i_1 and i_2 are restricted to integers. Put also

$$s = [(2 + n^{\sigma_2 + \sigma_1/3})/3], \quad t = [(1 + n^\tau)/2], \quad X = \exp(n^\beta) \quad \text{and} \quad \kappa = n^{3\delta}.$$

Assume first that $\delta_E \geq X^{-n/(st)^2}$. Then, if n is sufficiently large, all the conditions of Proposition 5.2 are satisfied because we have $1 \leq st \leq n$, $|\mathcal{A}| \geq s$, $|E| \geq s$, $c_{\mathcal{A}} \leq H(r)n^{\sigma_2}$ and $c_E \leq (|\xi| + |r|)n^{2\sigma_2}$, while the hypothesis on P implies

$$\max\{|P^{[j]}(\lambda \cdot x)|; \lambda \in \mathcal{A}, x \in E, 0 \leq j < 2t\} \leq \exp(-n^\nu) \leq X^{-\kappa n/(st)}.$$

In this case, Proposition 5.2 provides us with a non-zero polynomial $S \in \mathbb{Q}[T]$ and a point $x \in E$ satisfying

$$\deg(S) \leq \frac{5n}{st}, \quad H(S) \leq \exp\left(\frac{10n^\beta}{st}\right) \quad \text{and} \quad \frac{|S(x)|}{\text{cont}(S)} \leq \exp\left(-\frac{2n^{1+\beta+2\delta}}{s^2t^2}\right).$$

Write $x = i_1\xi + i_2r$ with $i_1, i_2 \in \mathbb{Z}$ and put $Q = \lambda_{i_1, i_2r}S$ so that $Q(\xi) = S(x)$. Then, Q is a non-zero polynomial of $\mathbb{Q}[T]$ and, using the crude estimates $1 \leq i_1 \leq n$ and $0 \leq i_2 \leq n$, Lemma 4.1 gives $\deg(Q) = \deg(S)$,

$$\frac{H(Q)}{H(S)} \leq (3nH(r))^{\deg(S)} \quad \text{and} \quad \frac{\text{cont}(S)}{\text{cont}(Q)} \leq (nH(r))^{\deg(S)}.$$

Since $\beta > 1$ and $st \leq n$, the last two quantities are bounded above by $\exp(n^\beta/(st)) \leq \exp(n^{1+\beta}/(st)^2)$ for n sufficiently large, and so the polynomial Q satisfies

$$\deg(Q) \leq \frac{5n}{st}, \quad H(Q) \leq \exp\left(\frac{11n^\beta}{st}\right) \quad \text{and} \quad \frac{|Q(\xi)|}{\text{cont}(Q)} \leq \exp\left(-\frac{n^{1+\beta+2\delta}}{s^2t^2}\right). \quad (19)$$

If $\delta_E < X^{-n/(st)^2}$, there exist integers i_1 and i_2 , both not 0 with absolute value at most n such that $|i_1\xi + i_2r| \leq \exp(-n^{1+\beta}/(st)^2)$. In this case, we define $Q(T) = (i_1T + i_2r)^{[n/(st)]}$. Since $n/(st) \geq n^{3\delta}$, this polynomial satisfies (19) if n is large enough. Thus, for each sufficiently large n , there exists a non-zero polynomial $Q \in \mathbb{Q}[T]$ satisfying (19). Since s and t behave like polynomials in n , this contradicts Gel'fond's Lemma 2.2. $\square$

Proof of Theorem 1.1(2). Suppose on the contrary that, for each sufficiently large integer n , the polynomial P_n satisfies

$$\max\{|P_n^{[j]}(r^i\xi)|; 0 \leq i \leq n^\sigma, 0 \leq j \leq n^\tau\} \leq \exp(-n^\nu), \quad (20)$$

and choose a positive real number δ such that $\nu \geq 1 + \beta - \sigma - \tau + 4\delta$. We claim that for any sufficiently large integer n , all the hypotheses of Proposition 5.2 are satisfied with

$$P = P_n, \quad s = [(1 + n^\sigma)/2], \quad t = [(1 + n^\tau)/2], \quad X = \exp(n^\beta), \quad \kappa = n^{3\delta},$$

$$A = \{1, r, \dots, r^{s-1}\}, \quad \mathcal{A} = \{\lambda_{a,0}; a \in A\} \quad \text{and} \quad E = \{\xi, r\xi, \dots, r^{s-1}\xi\}.$$

First of all, we have $\lambda \cdot x \in \{r^i\xi; 0 \leq i \leq n^\sigma\}$ for each $\lambda \in \mathcal{A}$ and each $x \in E$, and so the main condition (13) of this proposition follows from (20) and $X^{\kappa n/(st)} \leq \exp(n^\nu)$. We also find $c_A \leq H(r)^s$, $c_E \leq |\xi| \max(1, |r|)^s$ and $\delta_E \geq c_1^s$ where $c_1 = \min\{1, |r|, |r-1|, |\xi|\}$. Since $st \leq n$, this implies that 3^n , c_A^n , c_E^n and $\delta_E^{-s^2t^2/n}$ are all bounded above by $c_2^{n^s}$ for some constant $c_2 \geq 1$. As $\beta > 1 + \sigma$, this shows that the technical condition (12) is also satisfied if n is large enough. Then, Proposition 5.2 provides us with a non-zero polynomial $S \in \mathbb{Q}[T]$ and an integer i with $0 \leq i \leq s-1$ satisfying

$$\deg(S) \leq \frac{5n}{st}, \quad H(S) \leq \exp\left(\frac{10n^\beta}{st}\right) \quad \text{and} \quad \frac{|S(r^i\xi)|}{\text{cont}(S)} \leq \exp\left(-\frac{2n^{1+\beta+2\delta}}{s^2t^2}\right).$$

Put $Q(T) = S(r^iT) = \lambda_{r^i,0}S$, so that $Q(\xi) = S(r^i\xi)$. Then, Q is a non-zero polynomial of $\mathbb{Q}[T]$ and Lemma 4.1 gives $\deg(Q) = \deg(S)$,

$$\frac{H(Q)}{H(S)} \leq (3H(r)^s)^{\deg(S)} \quad \text{and} \quad \frac{\text{cont}(S)}{\text{cont}(Q)} \leq H(r)^{s \deg(S)}.$$

As $\beta > 1 + \sigma$ and $st \leq n$, these quantities are both bounded above by $\exp(n^\beta/(st)) \leq \exp(n^{1+\beta}/(st)^2)$ if n is sufficiently large, and then Q satisfies (19). Again this contradicts Gel'fond's Lemma 2.2. $\square$

6. Estimates for an Intersection

Throughout this section, we fix a positive integer s and we denote by $(\mathbf{e}_1, \dots, \mathbf{e}_s)$ the canonical basis of $\mathbb{Z}^s$. For each $\mathbf{x} \in \mathbb{Z}^s$ and each subset E of $\mathbb{Z}^s$, we define

$$\mathcal{O}(\mathbf{x}) = \{\mathbf{x} + \mathbf{e}_1, \dots, \mathbf{x} + \mathbf{e}_s\} \quad \text{and} \quad \mathcal{O}(E) = \bigcup_{i=1}^s (E + \mathbf{e}_i) = \bigcup_{\mathbf{x} \in E} \mathcal{O}(\mathbf{x}),$$

so that for subsets E and F of $\mathbb{Z}^s$, we have

$$E \subseteq (F - \mathbf{e}_1) \cap \cdots \cap (F - \mathbf{e}_s) \iff \mathcal{O}(E) \subseteq F. \quad (21)$$

We are interested here in the following type of result.

Proposition 6.1. *Let E and F be finite subsets of $\mathbb{Z}^s$ with $\mathcal{O}(E) \subseteq F$. Suppose that $|F| \leq s^2/4$. Then, we have $|F| \geq (s/2)|E|$. Moreover, if $\mathbf{x}_1, \dots, \mathbf{x}_r$ denote the distinct elements of E , there is a partition $F = F_1 \amalg \cdots \amalg F_r \amalg F_{r+1}$ of F such that, for each $i = 1, \dots, r$, we have $F_i \subseteq \mathcal{O}(\mathbf{x}_i)$ and $|F_i| \geq s/2$.*

Note that, for any given finite set E , the equivalent conditions (21) hold with $F = \mathcal{O}(E)$, and then we have $|F| \leq s|E|$. Thus any general estimate of the form $|F| \geq (s/c)|E|$ with a constant $c \geq 1$ is optimal up to the value of c . The first assertion of the proposition shows that we can take $c = 2$ when $|F| \leq s^2/4$. We will show that similar estimates hold in general when the cardinality of F is at most polynomial in s , with similar partitions of E and F into subsets of relatively small diameters. In Appendix B, we show that, for any pair of subsets E and F satisfying the slightly stronger condition $E \subseteq F \cap (F - \mathbf{e}_1) \cap \cdots \cap (F - \mathbf{e}_s)$, we have $|E| \leq (1/s)|F| \log |F|$, but the proof does not provide corresponding partitions for E and F .

Proof. We first observe that the differences $\mathbf{e}_i - \mathbf{e}_j$ with $1 \leq i, j \leq s$ and $i \neq j$ are all distinct and thus, for any pair of distinct points $\mathbf{x}, \mathbf{y}$ of $\mathbb{Z}^s$, the set $\mathcal{O}(\mathbf{x}) \cap \mathcal{O}(\mathbf{y})$ contains at most one element.

Define $F_i = \mathcal{O}(\mathbf{x}_i) \setminus (\mathcal{O}(\mathbf{x}_1) \cup \cdots \cup \mathcal{O}(\mathbf{x}_{i-1}))$ for $i = 1, \dots, r$, and let F_{r+1} denote the complement of $F_1 \cup \cdots \cup F_r$ in F . Then, $F_1, \dots, F_{r+1}$ form a partition of F with $F_i \subseteq \mathcal{O}(\mathbf{x}_i)$ for $i = 1, \dots, r$. By virtue of the preceding observation, we also have

$$|F_i| \geq |\mathcal{O}(\mathbf{x}_i)| - \sum_{j=1}^{i-1} |\mathcal{O}(\mathbf{x}_i) \cap \mathcal{O}(\mathbf{x}_j)| \geq s - (i-1) = s - i + 1$$

for each $i \leq r$. In particular, this gives $|F_i| \geq s/2$ for $i = 1, \dots, \min(r, \lfloor s/2 \rfloor + 1)$, and so $|F| \geq (s/2) \min(r, \lfloor s/2 \rfloor + 1)$. Since $|F| \leq s^2/4$, we conclude that $r \leq s/2$ and thus that $|F_i| \geq s/2$ for $i = 1, \dots, r$. $\square$

The statement of our main proposition requires additional notation. Given any point $\mathbf{x} = (x_1, \dots, x_s) \in \mathbb{Z}^s$, we write $\|\mathbf{x}\|_1$ to denote its ℓ_1 -norm $|x_1| + \cdots + |x_s|$. We also denote by U the subgroup of $\mathbb{Z}^s$ given by

$$U = \{(x_1, \dots, x_s) \in \mathbb{Z}^s; x_1 + \cdots + x_s = 0\}.$$

For each integer $k \geq 0$, we define

$$C_k = \{\mathbf{x} \in U; \|\mathbf{x}\|_1 \leq 2k\},$$

and observe, for later use, that any point of C_k has at most k positive coordinates and at most k negative coordinates. For any point $\mathbf{x} \in \mathbb{Z}^s$ and any integer $k \geq 0$, we also define

$$C_k(\mathbf{x}) = \mathbf{x} + C_k.$$

Proposition 6.2. *Let E and F be finite subsets of $\mathbb{Z}^s$ with $\mathcal{O}(E) \subseteq F$. Suppose that*

$$|F| \leq \frac{1}{2^{\ell+1}(\ell+1)!} \binom{s}{\ell+2} \quad (22)$$

for some integer ℓ with $0 \leq \ell \leq s-2$. Then, we have

$$|F| \geq \frac{s-\ell}{2(\ell+1)} |E|. \quad (23)$$

More precisely, if E is not empty, there exist an integer $r \geq 1$, a sequence of points $\mathbf{x}_1, \dots, \mathbf{x}_r$ of E , and partitions $E = E_1 \amalg \dots \amalg E_r$ and $F = F_1 \amalg \dots \amalg F_r \amalg F_{r+1}$ of E and F which, for $i = 1, \dots, r$, satisfy

$$(a) \ E_i \subseteq C_\ell(\mathbf{x}_i), \quad (b) \ F_i \subseteq \mathcal{O}(E_i), \quad (c) \ |F_i| \geq \frac{s-\ell}{2(\ell+1)} |E_i|.$$

The proof of this result requires three lemmas.

Lemma 6.3. *Let $k \geq 0$ be an integer, let C be a subset of C_k and let $D = \mathcal{O}(C)$. Then, we have $|D| \geq ((s-k)/(k+1))|C|$.*

Note that, since $\mathbf{e}_1 + C \subseteq D$, we also have $|D| \geq |C|$. Therefore, the conclusion of the lemma is interesting only when $k < s/2$.

Proof. For each point $(\mathbf{x}, i) \in C \times \{1, \dots, s\}$, we have either $\|\mathbf{x} + \mathbf{e}_i\|_1 = \|\mathbf{x}\|_1 - 1$ or $\|\mathbf{x} + \mathbf{e}_i\|_1 = \|\mathbf{x}\|_1 + 1$. Denote by N the set of points $(\mathbf{x}, i)$ in $C \times \{1, \dots, s\}$ which satisfy the first condition, and by P the set of those which satisfy the second condition. Since N and P form a partition of $C \times \{1, \dots, s\}$, we have

$$|N| + |P| = s|C|. \quad (24)$$

For any fixed $\mathbf{x} \in C$, the integers $i \in \{1, \dots, s\}$ such that $(\mathbf{x}, i) \in N$ are those for which the i th coordinate of $\mathbf{x}$ is negative. Since $C \subseteq C_k$, such a point $\mathbf{x}$ has at most k negative coordinates and therefore there are at most k values of i for which $(\mathbf{x}, i) \in N$. As this holds for any $\mathbf{x} \in C$, we deduce that

$$|N| \leq k|C|. \quad (25)$$

Consider now the surjective map $\varphi: C \times \{1, \dots, s\} \rightarrow D$ given by $\varphi(\mathbf{x}, i) = \mathbf{x} + \mathbf{e}_i$. For any $(\mathbf{x}, i) \in P$, the i th coordinate of $\varphi(\mathbf{x}, i)$ is positive. Since $D \subseteq \mathcal{O}(C_k)$, any point $\mathbf{y} \in D$ has at most $k+1$ positive coordinates, and therefore we get

$|P \cap \varphi^{-1}(\mathbf{y})| \leq k + 1$ for each $\mathbf{y} \in D$. The surjectivity of φ then implies

$$|P| = \sum_{\mathbf{y} \in D} |P \cap \varphi^{-1}(\mathbf{y})| \leq (k + 1)|D|. \quad (26)$$

The combination of (24)–(26) gives $(k + 1)|D| \geq |P| = s|C| - |N| \geq (s - k)|C|$, as announced. $\square$

For any integer $k \geq 0$, any point $\mathbf{x} \in \mathbb{Z}^s$ and any subset E of $\mathbb{Z}^s$, we define

$$C_k(\mathbf{x}, E) = C_k(\mathbf{x}) \cap E \quad \text{and} \quad D_k(\mathbf{x}, E) = \mathcal{O}(C_k(\mathbf{x}, E)).$$

With this notation, if a set F contains $\mathcal{O}(E)$, then it contains $D_k(\mathbf{x}, E)$ for any $k \geq 0$ and any $\mathbf{x} \in \mathbb{Z}^s$. We can now state the next lemma.

Lemma 6.4. *Let E be a finite subset of $\mathbb{Z}^s$. For any integer $k \geq 0$ and any point $\mathbf{x} \in \mathbb{Z}^s$, we have*

- (i) $|D_k(\mathbf{x}, E)| \geq \frac{s - k}{k + 1} |C_k(\mathbf{x}, E)|$,
- (ii) $|D_k(\mathbf{x}, E) \cap \mathcal{O}(E \setminus C_k(\mathbf{x}, E))| \leq (k + 1) |C_{k+1}(\mathbf{x}, E)|$.

Proof. Fix a choice of k and $\mathbf{x}$, and put $C = C_k(\mathbf{x}, E) - \mathbf{x}$ and $D = D_k(\mathbf{x}, E) - \mathbf{x}$. Then, C and D are subsets of $\mathbb{Z}^s$ with the same cardinality as $C_k(\mathbf{x}, E)$ and $D_k(\mathbf{x}, E)$ respectively. Since they satisfy the hypotheses $C \subseteq C_k$ and $D = \mathcal{O}(C)$ of Lemma 6.3, the inequality (i) follows directly from this lemma.

To prove (ii), it suffices to show that, for any $\mathbf{y} \in E \setminus C_k(\mathbf{x}, E)$ such that $D_k(\mathbf{x}, E) \cap \mathcal{O}(\mathbf{y}) \neq \emptyset$, we have $\mathbf{y} \in C_{k+1}(\mathbf{x}, E)$ and $|D_k(\mathbf{x}, E) \cap \mathcal{O}(\mathbf{y})| \leq k + 1$. Fix such a point $\mathbf{y}$, assuming that there exists at least one. Since $D_k(\mathbf{x}, E) \cap \mathcal{O}(\mathbf{y}) \neq \emptyset$, there is an integer $i \in \{1, \dots, s\}$ such that $\mathbf{y} + \mathbf{e}_i \in D_k(\mathbf{x}, E)$. For this choice of i , there is also a point $\mathbf{z} \in C_k(\mathbf{x}, E)$ and an integer $j \in \{1, \dots, s\}$ such that $\mathbf{y} + \mathbf{e}_i = \mathbf{z} + \mathbf{e}_j$. Rewriting this equality in the form

$$\mathbf{y} - \mathbf{x} = (\mathbf{z} - \mathbf{x}) + (\mathbf{e}_j - \mathbf{e}_i), \quad (27)$$

we deduce that $\|\mathbf{y} - \mathbf{x}\|_1 \leq \|\mathbf{z} - \mathbf{x}\|_1 + 2 \leq 2(k + 1)$ and also that $\mathbf{y} - \mathbf{x} \in U$ since U contains both $\mathbf{z} - \mathbf{x}$ and $\mathbf{e}_j - \mathbf{e}_i$. Since $\mathbf{y} \in E$, this shows that $\mathbf{y} \in C_{k+1}(\mathbf{x}, E)$. Moreover, since $\mathbf{y} \notin C_k(\mathbf{x}, E)$, we also have $\|\mathbf{y} - \mathbf{x}\|_1 > 2k$ and so $\|\mathbf{y} - \mathbf{x}\|_1 = 2k + 2$, because $\|\mathbf{y} - \mathbf{x}\|_1$ is an even integer. This observation combined with (27) and the fact that $\|\mathbf{z} - \mathbf{x}\|_1 \leq 2k$ tells us that the i th coordinate of $\mathbf{y} - \mathbf{x}$ is negative. As $\mathbf{y} - \mathbf{x}$ admits at most $k + 1$ negative coordinates, we deduce that there are at most $k + 1$ values of i such that $\mathbf{y} + \mathbf{e}_i \in D_k(\mathbf{x}, E)$, and so $|D_k(\mathbf{x}, E) \cap \mathcal{O}(\mathbf{y})| \leq k + 1$. $\square$

Lemma 6.5. *Let E , F and ℓ be as in the statement of Proposition 6.2. For each $\mathbf{x} \in E$, there exists at least one integer k with $0 \leq k \leq \ell$ such that*

$$|D_k(\mathbf{x}, E) \cap \mathcal{O}(E \setminus C_k(\mathbf{x}, E))| \leq \frac{s - k}{2(k + 1)} |C_k(\mathbf{x}, E)|. \quad (28)$$

Proof. Suppose on the contrary that there exists $\mathbf{x} \in E$ such that (28) does not hold for any k with $0 \leq k \leq \ell$. Using Lemma 6.4(ii), this gives

$$2(k+1)|C_{k+1}(\mathbf{x}, E)| \geq 2|D_k(\mathbf{x}, E) \cap \mathcal{O}(E \setminus C_k(\mathbf{x}, E))| > \frac{s-k}{k+1}|C_k(\mathbf{x}, E)|,$$

for $k = 0, \dots, \ell$. Multiplying these inequalities term by term for all these values of k and noting that $C_0(\mathbf{x}, E)$ is the singleton $\{\mathbf{x}\}$, we deduce that

$$2^{\ell+1}(\ell+1)!|C_{\ell+1}(\mathbf{x}, E)| > \binom{s}{\ell+1}.$$

Since F contains $D_{\ell+1}(\mathbf{x}, E)$, the above estimate combined with Lemma 6.4(i) leads to

$$|F| \geq |D_{\ell+1}(\mathbf{x}, E)| \geq \frac{s-\ell-1}{\ell+2}|C_{\ell+1}(\mathbf{x}, E)| > \frac{1}{2^{\ell+1}(\ell+1)!} \binom{s}{\ell+2},$$

against the hypothesis (22) of Proposition 6.2. $\square$

Proof of Proposition 6.2. Since the inequality (23) from the first assertion of the proposition follows from the estimates (c) of the second assertion, it suffices to prove the latter. To do so we proceed by induction on $|E|$. Let $\mathbf{x}_1 \in E$. Lemma 6.5 combined with Lemma 6.4(i) shows that there exists an integer k with $0 \leq k \leq \ell$ such that, upon putting $E_1 = C_k(\mathbf{x}_1, E)$ and $F_1 = D_k(\mathbf{x}_1, E) \setminus \mathcal{O}(E \setminus E_1)$, we have

$$\begin{aligned} |F_1| &= |D_k(\mathbf{x}_1, E)| - |D_k(\mathbf{x}_1, E) \cap \mathcal{O}(E \setminus E_1)| \\ &\geq \frac{s-k}{k+1}|C_k(\mathbf{x}_1, E)| - \frac{s-k}{2(k+1)}|C_k(\mathbf{x}_1, E)| \\ &= \frac{s-k}{2(k+1)}|E_1| \geq \frac{s-\ell}{2(\ell+1)}|E_1|. \end{aligned}$$

Therefore the sets E_1 and F_1 fulfill the conditions (a)–(c) of Proposition 6.2 for $i = 1$.

If $E = E_1$, this proves the proposition with $r = 1$ and $F_2 = F \setminus F_1$. In particular, the proposition is verified when $|E| = 1$. Assume therefore that $E \neq E_1$. We put $E' = E \setminus E_1$ and $F' = F \setminus F_1$. By construction, F_1 and $\mathcal{O}(E')$ are disjoint sets. Since $\mathcal{O}(E') \subseteq \mathcal{O}(E) \subseteq F$, this implies that $\mathcal{O}(E') \subseteq F'$. Thus the hypotheses of Proposition 6.2 are also satisfied by E' and F' instead of E and F , with the same value of ℓ . Since $|E'| < |E|$, we may assume by induction that there exists an integer $r \geq 2$, a sequence of points $\mathbf{x}_2, \dots, \mathbf{x}_r$ of E' and partitions $E' = E_2 \amalg \dots \amalg E_r$ and $F' = F_2 \amalg \dots \amalg F_{r+1}$ which fulfill the conditions (a)–(c) of the proposition for $i = 2, \dots, r$. Then the partitions $E = E_1 \amalg \dots \amalg E_r$ and $F = F_1 \amalg \dots \amalg F_{r+1}$ have all the required properties. $\square$

7. Estimates for the gcd

We say that a finite subset A of $\mathbb{Q}_+$ with s elements is multiplicatively independent if it generates a free subgroup of $\mathbb{Q}_+$ of rank s . This happens for example when A consists of s prime numbers. The main result of this section is the following statement which immediately implies Theorem 1.2.

Theorem 7.1. *Let A be a finite multiplicatively independent subset of $\mathbb{Q}_+$, let s be its cardinality, let P be a polynomial of $\mathbb{Q}[T]$ with $P(0) \neq 0$, and let $Q = \gcd\{P(aT); a \in A\}$. Suppose that the number of non-associate irreducible factors of P is at most*

$$N(s, \ell) := \binom{s}{\ell+2} \frac{1}{2^{\ell+1}(\ell+1)!}$$

for some integer ℓ with $0 \leq \ell \leq s-2$. Then, we have

$$\deg(Q) \leq \frac{2(\ell+1)}{s-\ell} \deg(P), \quad \log H(Q) \leq \frac{2(\ell+1)}{s-\ell} (\log H(P) + c_1 \deg(P)) \quad (29)$$

where $c_1 = 8 + (4\ell+1) \log(c_A)$ and $c_A = \max_{a \in A} H(a)$.

The proof of the theorem proceeds first by a reduction to a specific type of polynomial P . To state and prove the lemma that we apply for this purpose, we use the following notation.

For each $a \in \mathbb{Q}_+$, we simply write λ_a to denote the automorphism $\lambda_{a,0}$ of $\mathbb{Q}[T]$ which maps a polynomial $P \in \mathbb{Q}[T]$ to $(\lambda_a P)(T) = P(aT)$ (see Sec. 4). Moreover, for a given subgroup G of $\mathbb{Q}_+$, we say that two polynomials P_1 and P_2 of $\mathbb{Q}[T]$ are G -equivalent and write $P_1 \sim_G P_2$ if there exists $a \in G$ such that $P_2 = \lambda_a(P_1)$. We also say that a polynomial $P \in \mathbb{Q}[T]$ is G -pure if it can be written as a product of G -equivalent irreducible polynomials of $\mathbb{Q}[T]$.

Lemma 7.2. *Let G be a subgroup of $\mathbb{Q}_+$, let A be a finite subset of G , let P be a non-zero polynomial of $\mathbb{Q}[T]$, and let $Q = \gcd\{P(aT); a \in A\}$. Then, we can write P as a product $P = P_1 \cdots P_N$ of G -pure polynomials $P_1, \dots, P_N$ with simple roots so that $Q = \prod_{i=1}^N \gcd\{P_i(aT); a \in A\}$.*

Proof. We first observe that P can be written as a product $P = P_1 \cdots P_M$ of polynomials $P_1, \dots, P_M$ with simple roots such that P_{i+1} divides P_i for $i = 1, \dots, M-1$, and that such a factorization is unique up to multiplication of each P_i by an element of $\mathbb{Q}^*$. For each $a \in A$, the equality $\lambda_a P = (\lambda_a P_1) \cdots (\lambda_a P_M)$ provides a factorization of $\lambda_a P$ of the same type. From this we deduce that $Q = \prod_{i=1}^M \gcd\{P_i(aT); a \in A\}$ is the corresponding factorization of Q . This reduces the proof of Lemma 7.2 to the case where P has no multiple roots.

Let $R_1, \dots, R_L$ be a set of representatives for the equivalence classes of G -equivalent irreducible factors of P . We can also write P as a product $P = P_1 \cdots P_L$ of G -pure polynomials $P_1, \dots, P_L$ such that, for each $i = 1, \dots, L$, all irreducible

factors of P_i are G -equivalent to R_i . Again, such a factorization is unique up to multiplication of each P_i by an element of $\mathbb{Q}^*$. Moreover, for each $a \in A$, the corresponding factorization of $\lambda_a P$ is $\lambda_a P = (\lambda_a P_1) \cdots (\lambda_a P_L)$, and so we deduce that $Q = \prod_{i=1}^L \gcd\{P_i(aT); a \in A\}$. This further reduces the proof of Lemma 7.2 to the case where P is G -pure and so completes the proof of the lemma. $\square$

Proof of Theorem 7.1. Let G denote the subgroup of $\mathbb{Q}_+$ generated by A . We claim that the conclusion (29) of the theorem holds with the constant $c_2 = c_1 - 2$ instead of c_1 when P is G -pure with no multiple factors. If we take this for granted and apply it to each factor in the factorization $P = P_1 \cdots P_N$ of P provided by Lemma 7.2, we find that for each i the polynomial $Q_i = \gcd\{P_i(aT); a \in A\}$ satisfies

$$\deg(Q_i) \leq \rho \deg(P_i) \quad \text{and} \quad \log H(Q_i) \leq \rho(c_2 \deg(P_i) + \log H(P_i))$$

where $\rho = 2(\ell + 1)/(s - \ell)$. Since Lemma 7.2 gives $Q = Q_1 \cdots Q_N$, these inequalities in turn imply that $\deg(Q) \leq \rho \deg(P)$ and that

$$\begin{aligned} \log H(Q) &\leq \deg(Q) + \sum_{i=1}^N \log H(Q_i) \\ &\leq \rho \sum_{i=1}^N ((c_2 + 1) \deg(P_i) + \log H(P_i)) \\ &\leq \rho((c_2 + 2) \deg(P) + \log H(P)), \end{aligned}$$

as announced.

In order to prove our claim, we now assume that P is G -pure with no multiple factors. Without loss of generality, we may further assume that Q is non-constant. Write $A = \{a_1, \dots, a_s\}$, and for each $\mathbf{x} = (x_1, \dots, x_s) \in \mathbb{Z}^s$ define $a^{\mathbf{x}} := a_1^{x_1} \cdots a_s^{x_s}$. With this notation, we have $a_i = a^{\mathbf{e}_i}$ for $i = 1, \dots, s$ where $(\mathbf{e}_1, \dots, \mathbf{e}_s)$ denotes the canonical basis of $\mathbb{Z}^s$. Moreover, since $a_1, \dots, a_s$ are multiplicatively independent, the map from $\mathbb{Z}^s$ to G which sends each $\mathbf{x} \in \mathbb{Z}^s$ to $a^{\mathbf{x}} \in G$ is a group isomorphism. Choose an irreducible factor R of P . As $P(0) \neq 0$, the polynomial R is not a rational multiple of T , and so Lemma 4.2 shows that, for distinct points $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^s$, the translates $R(a^{-\mathbf{x}}T)$ and $R(a^{-\mathbf{y}}T)$ are not associates. Therefore P is an associate of $\prod_{\mathbf{x} \in F} R(a^{-\mathbf{x}}T)$ for a unique finite subset F of $\mathbb{Z}^s$, and we find

$$Q = \gcd \left\{ \prod_{\mathbf{x} \in F} R(a^{-\mathbf{x}} a_i T); i = 1, \dots, s \right\} = \prod_{\mathbf{x} \in E} R(a^{-\mathbf{x}} T),$$

where $E = (F - \mathbf{e}_1) \cap \cdots \cap (F - \mathbf{e}_s)$. We now apply Proposition 6.2 to the sets E and F . Since Q is non-constant, the set E is not empty and this proposition provides an integer $r \geq 1$, a sequence of points $\mathbf{x}_1, \dots, \mathbf{x}_r$ of E , and partitions

$$E = E_1 \amalg \cdots \amalg E_r \quad \text{and} \quad F = F_1 \amalg \cdots \amalg F_r \amalg F_{r+1}$$

of E and F which, for $i = 1, \dots, r$, satisfy $E_i \subseteq C_\ell(\mathbf{x}_i)$, $F_i \subseteq \mathcal{O}(E_i)$ and $|E_i| \leq \rho|F_i|$ where $\rho = 2(\ell + 1)/(s - \ell)$. The third set of conditions implies $|E| \leq \rho|F|$. Since for each $\mathbf{x} \in \mathbb{Z}^s$ the polynomial $R(a^{-\mathbf{x}}T)$ has the same degree as R , we deduce that

$$\deg(Q) = |E| \deg(R) \leq \rho|F| \deg(R) = \rho \deg(P).$$

To compare the heights of Q and P , we put $R_i = R(a^{-\mathbf{x}_i}T)$ for $i = 1, \dots, r$. The condition $E_i \subseteq C_\ell(\mathbf{x}_i)$ implies that, for each $\mathbf{x} \in E_i$, we have $\|\mathbf{x} - \mathbf{x}_i\|_1 \leq 2\ell$ and so

$$\log H(a^{\mathbf{x} - \mathbf{x}_i}) \leq 2\ell \max_{1 \leq k \leq s} \log H(a_k) = 2\ell \log(c_A).$$

Since $R(a^{-\mathbf{x}}T) = R_i(a^{\mathbf{x}_i - \mathbf{x}}T)$, Lemma 4.1 then gives

$$\begin{aligned} |\log H(R(a^{-\mathbf{x}}T)) - \log H(R_i)| &\leq \log(3H(a^{\mathbf{x} - \mathbf{x}_i})) \deg(R) \\ &\leq (2 + 2\ell \log(c_A)) \deg(R) \end{aligned}$$

for each $\mathbf{x} \in E_i$. Since $Q = \prod_{i=1}^r \prod_{\mathbf{x} \in E_i} R(a^{-\mathbf{x}}T)$, we deduce that

$$\begin{aligned} \left| \log H(Q) - \sum_{i=1}^r |E_i| \log H(R_i) \right| &\leq \deg(Q) + \sum_{i=1}^r \sum_{\mathbf{x} \in E_i} |\log H(R(a^{-\mathbf{x}}T)) - \log H(R_i)| \\ &\leq \deg(Q) + (2 + 2\ell \log(c_A)) |E| \deg(R) \\ &\leq (3 + 2\ell \log(c_A)) \deg(Q). \end{aligned}$$

The condition $F_i \subseteq \mathcal{O}(E_i)$ in turn implies that, for each $\mathbf{x} \in F_i$, we have $\|\mathbf{x} - \mathbf{x}_i\|_1 \leq 2\ell + 1$ and so the same computations lead to

$$\left| \log H(P) - \sum_{i=1}^{r+1} |F_i| \log H(R_i) \right| \leq (3 + (2\ell + 1) \log(c_A)) \deg(P).$$

Putting all these estimates together we conclude finally that

$$\begin{aligned} \log H(Q) &\leq (3 + 2\ell \log(c_A)) \deg(Q) + \sum_{i=1}^r |E_i| \log H(R_i) \\ &\leq \rho \left((3 + 2\ell \log(c_A)) \deg(P) + \sum_{i=1}^{r+1} |F_i| \log H(R_i) \right) \\ &\leq \rho(c_2 \deg(P) + \log H(P)) \end{aligned}$$

where $c_2 = 6 + (4\ell + 1) \log(c_A)$. □

8. Further Small Value Estimates

The next result refines Proposition 5.2 in a context where the estimates of the preceding section apply. We use it below to prove Theorem 1.1(5) in a general form involving a subgroup of arbitrary rank.

Proposition 8.1. *Let $\ell \geq 0$ and $n, t \geq 1$ be integers. Let A be a finite multiplicatively independent subset of $\mathbb{Q}_+$, let $s = |A|$ denote its cardinality, and let E be a*

finite non-empty subset of $\mathbb{C}^\times$. Assume that

$$s \geq \max\{\ell + 2, 2\ell\}, \quad \max(s, |E|)t \leq n \leq N(s, \ell) := \binom{s}{\ell + 2} \frac{1}{(\ell + 1)!2^{\ell+1}}. \quad (30)$$

Finally, let X be a real number satisfying

$$X^\epsilon \geq \max\{3^n, c_A^n, (2 + c_E)^n, \delta_E^{-|E|^2 t^2/n}\}, \quad (31)$$

where $\epsilon = (4\ell + 10)^{-1}$, $c_A = \max_{a \in A} H(a)$ and $c_E = \max_{\xi \in E} \max\{|\xi|, |\xi|^{-1}\}$. Suppose that there exists a non-zero polynomial P of $\mathbb{Z}[T]$ of degree at most n and height at most X satisfying

$$\max\{|P^{[j]}(a\xi)|; a \in A, \xi \in E, 0 \leq j < 2t\} \leq X^{-\kappa n/(|E|t)} \quad (32)$$

for some real number $\kappa > 2 + 34\epsilon$. Then, there exists a primary polynomial $S \in \mathbb{Z}[T]$ with

$$\deg(S) \leq \frac{2n}{st}, \quad H(S) \leq X^{4/(st)} \quad \text{and} \quad \prod_{\xi \in E} |S(\xi)| \leq X^{-\kappa' n/t^2}, \quad (33)$$

where $\kappa' = (\kappa - 2 - 34\epsilon)/(64(\ell + 1))$.

In the applications that we will make of this result, the cardinality s of A is bounded below by n^σ for some real number $\sigma > 0$, and so the condition $n \leq N(s, \ell)$ is satisfied with $\ell = \lceil 1/\sigma \rceil$ provided that n is large enough.

Proof. Write P in the form $P(T) = T^m \tilde{P}(T)$ where $\tilde{P}(T) \in \mathbb{Z}[T]$ is not divisible by T . Then, $\tilde{P}$ also has degree at most n and height at most X . Moreover, for any $a \in A$, any $\xi \in E$ and any integer j with $0 \leq j < 2t$, we find

$$\begin{aligned} |\tilde{P}^{[j]}(a\xi)| &= \left| \sum_{h=0}^j (-1)^h \binom{m+h-1}{h} (a\xi)^{-m-h} P^{[j-h]}(a\xi) \right| \\ &\leq \sum_{h=0}^j 2^{m+h-1} \max(1, H(a)|\xi|^{-1})^{m+2t} \max_{0 \leq i < 2t} |P^{[i]}(a\xi)| \\ &\leq (2c_A c_E)^{m+2t} X^{-\kappa n/(t|E|)}. \end{aligned}$$

Since $n \geq t|E|$ and $(2c_A c_E)^{m+2t} \leq (2c_A c_E)^{3n} \leq X^{9\epsilon}$, we conclude that

$$\max\{|\tilde{P}^{[j]}(a\xi)|; a \in A, \xi \in E, 0 \leq j < 2t\} \leq X^{-(\kappa-9\epsilon)n/(t|E|)}. \quad (34)$$

Let $\tilde{Q}$ be the greatest common divisor in $\mathbb{Q}[T]$ of the polynomials $\tilde{P}(aT)$ with $a \in A$. Since A is a multiplicatively independent subset of $\mathbb{Q}_+$, since $\tilde{P}(0) \neq 0$, and since the number of irreducible factors of $\tilde{P}$ is at most $\deg(\tilde{P}) \leq n \leq N(s, \ell)$, Theorem 7.1 gives

$$\deg(\tilde{Q}) \leq \frac{2(\ell + 1)}{s - \ell} \deg(\tilde{P}) \leq 4(\ell + 1) \frac{n}{s}$$

and

$$\begin{aligned} \log H(\tilde{Q}) &\leq \frac{2(\ell+1)}{s-\ell} (\log H(\tilde{P}) + (8 + (4\ell+1)\log(c_A)) \deg(\tilde{P})) \\ &\leq \frac{4(\ell+1)}{s} (\log(X) + 8n + (4\ell+1)\log(c_A)n) \\ &\leq 4(\ell+1)(2-\epsilon) \frac{\log X}{s} \end{aligned}$$

where the last estimation uses $\max(1, \log c_A)n \leq \epsilon \log X$ and $(4\ell+10)\epsilon = 1$.

Let Q be the greatest common divisor in $\mathbb{Q}[T]$ of the family of polynomials $\tilde{P}^{[j]}(aT) = (\lambda_{a,0}\tilde{P}^{[j]})(T)$ with $a \in A$ and $0 \leq j < t$. Since Q divides $\tilde{Q}$, we have

$$\deg(Q) \leq 4(\ell+1) \frac{n}{s} \quad \text{and} \quad \log H(Q) \leq \deg(\tilde{Q}) + \log H(\tilde{Q}) \leq 8(\ell+1) \frac{\log X}{s}.$$

Moreover, Proposition 5.1 applied to $\tilde{P}$ gives

$$\prod_{\xi \in E} \left(\frac{|Q(\xi)|}{\text{cont}(Q)} \right)^t \leq X^{25\epsilon n} H(\tilde{P})^{2n} X^{-(\kappa-9\epsilon)n} \leq X^{-(\kappa-2-34\epsilon)n} \leq X^{-64(\ell+1)\kappa'n}.$$

This means that Q satisfies the hypotheses of Lemma 2.1 with $\rho = 8(\ell+1)/s$ and $c = 32(\ell+1)\kappa'$. Consequently, there is at least one irreducible factor R of Q in $\mathbb{Q}[T]$ which satisfies

$$\prod_{\xi \in E} \left(\frac{|R(\xi)|}{\text{cont}(R)} \right)^t \leq (X^{\deg(R)} H(R)^n)^{-2\kappa's}. \quad (35)$$

By Lemma 4.3, this polynomial also satisfies

$$\deg(R) \leq \frac{n}{st} \quad \text{and} \quad H(R) \leq ((3c_A)^{2n} H(\tilde{P}))^{1/(st)} \leq X^{(1+4\epsilon)/(st)} \leq X^{2/(st)}.$$

Applying Lemma 2.1 to R with $\rho = 2/(st)$ and $c = 2\rho\kappa's = 4\kappa'/t$, we deduce that some power S of R satisfies

$$\deg(S) \leq \frac{2n}{st}, \quad H(S) \leq X^{4/(st)} \quad \text{and} \quad \prod_{\xi \in E} \left(\frac{|S(\xi)|}{\text{cont}(S)} \right)^t \leq X^{-\kappa'n/t}.$$

The quotient of S by its content is then a (non-constant) primary polynomial of $\mathbb{Z}[T]$ with the required properties (33). $\square$

For $m = 2$, the following result reduces to Theorem 1.1(5).

Theorem 8.2. *Let $\xi_1, \dots, \xi_m$ be $\mathbb{Q}$ -linearly independent complex numbers which generate a field of transcendence degree 1 over $\mathbb{Q}$. Let $\beta, \sigma, \tau, \nu \in \mathbb{R}$ with*

$$\sigma \geq 0, \quad \tau \geq 0, \quad \beta > 1 > \frac{3m\sigma}{m+2} + \tau \quad \text{and} \quad \nu > 1 + \beta - \frac{2m\sigma}{m+2} - \tau.$$

Then, for infinitely many integers $n \geq 1$, there is no non-zero polynomial $P \in \mathbb{Z}[T]$ of degree at most n and height at most $\exp(n^\beta)$ which satisfies

$$|P^{[j]}(i_1\xi_1 + \cdots + i_m\xi_m)| \leq \exp(-n^\nu)$$

for each choice of integers $i_1, \dots, i_m, j$ with $0 \leq i_1, \dots, i_m \leq n^\sigma$ and $0 \leq j < n^\tau$.

Proof. Suppose on the contrary that such a polynomial exists for each sufficiently large value of n . Then we have $\sigma > 0$ by [7, Proposition 1]. Moreover, since $\xi_1, \dots, \xi_m$ are not all algebraic over $\mathbb{Q}$, we may assume without loss of generality that ξ_1 is transcendental over $\mathbb{Q}$. Define

$$\lambda = \frac{\sigma}{m+2}, \quad \ell = \left\lceil \frac{2}{m\lambda} \right\rceil,$$

$$\delta = \frac{1}{8} \min\{4m\lambda, 1 - 3m\lambda - \tau, \nu - 1 - \beta + 2m\lambda + \tau\},$$

and note that the hypotheses lead to $\delta > 0$.

For a given positive integer n , define A to be the set of all prime numbers p with $p \leq n^{m\lambda}$, and define E to be the set of all linear combinations $i_1\xi_1 + \cdots + i_m\xi_m$ with integer coefficients in the range $1 \leq i_1, \dots, i_m \leq n^{2\lambda}$, which are not algebraic over $\mathbb{Q}$ and have absolute value at least 1. Since for fixed $i_2, \dots, i_m$, there are at most $1 + 2/|\xi_1|$ values of i_1 for which $i_1\xi_1 + \cdots + i_m\xi_m$ has absolute value less than one, and at most one value of i_1 for which it is algebraic over $\mathbb{Q}$, we readily get that, for n sufficiently large, we have

$$n^{m\lambda-\delta} \leq |A| \leq n^{m\lambda} \quad \text{and} \quad n^{2m\lambda-\delta} \leq |E| \leq n^{2m\lambda}.$$

Suppose first that $\delta_E \geq \exp(-n^{1+\beta-4m\lambda-2\tau-\delta})$. Then, we claim that, if n is sufficiently large, all the hypotheses of Proposition 8.1 are satisfied with the choice of

$$s = |A|, \quad t = [(1 + n^\tau)/2], \quad X = \exp(n^\beta) \quad \text{and} \quad \kappa = n^{6\delta}.$$

First of all, we have $\max(s, |E|)t \leq n$ because $2m\lambda + \tau < 1$. As $\delta \leq m\lambda/2$, we have $s \geq n^{m\lambda/2}$ and so $N(s, \ell) \geq n$ if n is large enough. For large n , we also find $c_A \leq n$, $c_E \leq n$ and $\delta_E^{-|E|^2 t^2/n} \leq \exp(n^{\beta-\delta})$. Thus (30) and (31) hold, while the hypothesis on P gives

$$\max\{|P^{[j]}(a\xi)|; a \in A, \xi \in E, 0 \leq j < 2t\} \leq \exp(-n^\nu) \leq X^{-\kappa n/(t|E|)}.$$

Consequently, for each sufficiently large value of n , there exists $S \in \mathbb{Z}[T] \setminus \{0\}$ with

$$\deg(S) \leq \frac{2n}{st}, \quad H(S) \leq \exp\left(\frac{4n^\beta}{st}\right) \quad \text{and} \quad \prod_{\xi \in E} |S(\xi)| \leq \exp(-n^{1+\beta-2\tau+5\delta}).$$

Since $|E| \leq n^{2m\lambda}$, the last condition implies the existence of a point $\xi \in E$ such that

$$|S(\xi)| \leq \exp(-n^{1+\beta-2m\lambda-2\tau+5\delta}).$$

Moreover, since ξ is transcendental over $\mathbb{Q}$, we have $S(\xi) \neq 0$. Define

$$Q(T_1, \dots, T_m) = S(i_1 T_1 + \dots + i_m T_m) \in \mathbb{Z}[T_1, \dots, T_m],$$

where $i_1, \dots, i_m$ are the positive integers for which $\xi = i_1 \xi_1 + \dots + i_m \xi_m$. Since $i_1, \dots, i_m$ are bounded above by $n^{2\lambda}$, we find, assuming that n is sufficiently large,

$$\begin{aligned} \deg(Q) &\leq \deg(S) \leq n^{1-m\lambda-\tau+2\delta}, \\ H(Q) &\leq (1 + mn^{2\lambda})^{\deg(S)} H(S) \leq \exp(n^{\beta-m\lambda-\tau+2\delta}), \\ 0 &< |Q(\xi_1, \dots, \xi_m)| = |S(\xi)| \leq \exp(-n^{1+\beta-2m\lambda-2\tau+5\delta}). \end{aligned}$$

Suppose now that $\delta_E < \exp(-n^{1+\beta-4m\lambda-2\tau-\delta})$, and choose integers $i_1, \dots, i_m$ not all zero, in absolute value at most $n^{2\lambda}$, such that $|i_1 \xi_1 + \dots + i_m \xi_m| = \delta_E$. Then, if n is large enough, the polynomial $Q = (i_1 T_1 + \dots + i_m T_m)^{[2n/(st)]}$ satisfies the same final estimates as in the preceding case, because $\delta_E^{[2n/(st)]} \leq \exp(-n^{2+\beta-5m\lambda-3\tau-\delta}) \leq \exp(-n^{1+\beta-2m\lambda-2\tau+7\delta})$. The existence of such a polynomial Q for each large enough n contradicts Lemma 2.2. $\square$

9. A Note on the Zarankiewicz Problem

Given integers m_1, n_1, m, n with $2 \leq m_1 \leq m$ and $2 \leq n_1 \leq n$, a well-known problem of Zarankiewicz asks for the smallest integer $k = k(m_1, n_1; m, n)$ such that any $m \times n$ matrix with coefficients in $\{0, 1\}$ containing at least k ones admits a sub-matrix of size $m_1 \times n_1$ consisting only of ones. Chapter 12 of the book by Erdős and Spencer [3] provides general estimates for this quantity along with references to early work on this problem. In particular, we mention a result of Kövari, Sós and Turán [6] which shows that $k(2, 2; n, n) = n^{3/2}(1 - o(1))$. In the next section, we will use the following result which we view as an estimate for a continuous version of Zarankiewicz problem in the case $m_1 = 2$.

Proposition 9.1. *Let A and E be finite non-empty sets, let κ_1 and κ_2 be positive real numbers, and let $\varphi: A \times E \rightarrow [0, \kappa_1]$ be any function on $A \times E$ with values in the interval $[0, \kappa_1]$. Suppose that the inequality*

$$\sum_{\xi \in E} \min\{\varphi(a_1, \xi), \varphi(a_2, \xi)\} \leq \kappa_2$$

holds for any pair of distinct elements a_1 and a_2 of A . Then, we have

$$\sum_{a \in A} \sum_{\xi \in E} \varphi(a, \xi) \leq \kappa_1 |E| + \kappa_2 \binom{|A|}{2}.$$

This gives $k(2, n_1; m, n) \leq 1 + n + (n_1 - 1)m(m - 1)/2$ in connection to the problem of Zarankiewicz mentioned above. Indeed, an $m \times n$ matrix with coefficients in $\{0, 1\}$ can be viewed as a function $\varphi: A \times E \rightarrow \{0, 1\}$ where $A = \{1, \dots, m\}$ and $E = \{1, \dots, n\}$. If it contains no $2 \times n_1$ sub-matrix consisting entirely of ones, the

hypotheses of the proposition are satisfied with $\kappa_1 = 1$ and $\kappa_2 = n_1 - 1$ and consequently the matrix contains at most $n + (n_1 - 1)m(m - 1)/2$ ones.

Proof. Let $[0, \infty)^E$ denote the set of all functions from E to $[0, \infty)$. We first observe that, if ϕ and ψ belong to this set then their minimum, their maximum and their sum

$$\phi + \psi = \min(\phi, \psi) + \max(\phi, \psi) \quad (36)$$

also belong to it. Moreover, the ℓ_1 -norm of any function $\phi: E \rightarrow [0, \infty)$ is simply $\|\phi\|_1 = \sum_{\xi \in E} \phi(\xi)$. Therefore the ℓ_1 -norm is additive on $[0, \infty)^E$ and by applying it on both sides of the equality (36) with functions $\phi, \psi \in [0, \infty)^E$, we obtain

$$\|\phi\|_1 + \|\psi\|_1 = \|\min\{\phi, \psi\}\|_1 + \|\max\{\phi, \psi\}\|_1. \quad (37)$$

Let $m = |A|$ and let $a_1, \dots, a_m$ denote the m elements of A . For $i = 1, \dots, m$, we define a function $\phi_i: E \rightarrow [0, \kappa_1]$ by putting $\phi_i(\xi) = \varphi(a_i, \xi)$ for each $\xi \in E$. By hypothesis, we have $\|\min(\phi_i, \phi_j)\|_1 \leq \kappa_2$ for any pair of integers i and j with $1 \leq i < j \leq m$. So, for $j = 2, \dots, m$, the function $\min(\max\{\phi_1, \dots, \phi_{j-1}\}, \phi_j) = \max\{\min(\phi_1, \phi_j), \dots, \min(\phi_{j-1}, \phi_j)\}$ satisfies

$$\|\min(\max\{\phi_1, \dots, \phi_{j-1}\}, \phi_j)\|_1 \leq \sum_{i=1}^{j-1} \|\min(\phi_i, \phi_j)\|_1 \leq (j-1)\kappa_2.$$

Applying (37) with $\phi = \max\{\phi_1, \dots, \phi_{j-1}\}$ and $\psi = \phi_j$, we deduce that

$$\|\max\{\phi_1, \dots, \phi_{j-1}\}\|_1 + \|\phi_j\|_1 \leq (j-1)\kappa_2 + \|\max\{\phi_1, \dots, \phi_j\}\|_1.$$

Summing these inequalities term by term for $j = 2, \dots, m$, we obtain after simplification

$$\sum_{j=1}^m \|\phi_j\|_1 \leq \sum_{j=2}^m (j-1)\kappa_2 + \|\max\{\phi_1, \dots, \phi_m\}\|_1.$$

Since $\max\{\phi_1, \dots, \phi_m\}$ takes values in $[0, \kappa_1]$, its ℓ_1 -norm is at most $\kappa_1|E|$, and the conclusion follows upon noting that $\sum_{a \in A} \sum_{\xi \in E} \varphi(a, \xi) = \sum_{j=1}^m \|\phi_j\|_1$. $\square$

10. Values of Polynomials at Multiples of ξ

In this section, we first prove Theorem 1.3 as a consequence of the next proposition, and then proceed with the proof of Theorem 1.1(3).

Proposition 10.1. *Let $n \in \mathbb{N}^*$, let A be a finite subset of $\mathbb{Q}_+$, and let E be a finite subset of $\mathbb{C}^\times$. Assume that*

$$|A| \geq 2 \quad \text{and} \quad \binom{|A|}{2} \leq |E| \leq n. \quad (38)$$

Finally, let ϵ and X be real numbers with

$$0 < \epsilon \leq \frac{1}{10} \quad \text{and} \quad X^\epsilon \geq \max\{e^n, c_A^n, (2 + c_E)^n, \Delta_E^{-1/n}\}, \quad (39)$$

where $c_A = \max_{a \in A} H(a)$ and $c_E = \max_{\xi \in E} \max(|\xi|, |\xi|^{-1})$. Suppose that there exists a non-zero polynomial P of $\mathbb{Z}[T]$ of degree at most n and height at most X satisfying

$$\prod_{a \in A} \prod_{\xi \in E} |P(a\xi)| < X^{-16\kappa|E|n} \quad (40)$$

for some real number $\kappa \geq 6$. Then, there exist a primary polynomial $S \in \mathbb{Q}[T]$ and a point $\xi \in E$ satisfying

$$\deg(S) \leq n, \quad H(S) \leq X^{2+2\epsilon} \quad \text{and} \quad \frac{|S(\xi)|}{\|S\|} \leq X^{-\kappa n}.$$

Proof. Applying Lemma 2.1 with $\rho = 1$ and $c = 8\kappa|E|$, we find that there exists a power Q of some non-constant irreducible factor of P in $\mathbb{Z}[T]$ satisfying

$$\deg(Q) \leq n, \quad H(Q) \leq X^2 \quad \text{and} \quad \prod_{a \in A} \prod_{\xi \in E} |Q(a\xi)| \leq X^{-2\kappa|E|n}. \quad (41)$$

We also note that Q is not a power of T because, for each $a \in A$ and each $\xi \in E$, we have $|a| \geq c_A^{-1} \geq X^{-\epsilon/n}$ and $|\xi| \geq c_E^{-1} \geq X^{-\epsilon/n}$ and so, for a power of T , the product $\prod_{a \in A} \prod_{\xi \in E} |Q(a\xi)|$ would be bounded below by $X^{-2\epsilon|A||E|} \geq X^{-4\epsilon|E|n}$ against the upper bound. For each $a \in A$ and each $\xi \in E$, we find

$$\|Q(aT)\| \geq H(a)^{-n} \|Q\| \geq c_A^{-n} \geq X^{-\epsilon}, \quad \frac{|Q(a\xi)|}{\|Q(aT)\|} \leq (|\xi| + 1)^n \leq (c_E + 1)^n \leq X^\epsilon.$$

Therefore we can write

$$\min(1, |Q(a\xi)|) \geq \frac{|Q(a\xi)|}{X^\epsilon \|Q(aT)\|} = X^{-\varphi(a, \xi)n} \quad (42)$$

for some real number $\varphi(a, \xi) \geq 0$. This defines a function $\varphi: A \times E \rightarrow [0, \infty)$ which, by the last condition in (41), satisfies

$$\sum_{a \in A} \sum_{\xi \in E} \varphi(a, \xi) \geq 2\kappa|E|. \quad (43)$$

Moreover, for each $a \in A$, Lemma 4.1 gives

$$H(Q(aT)) \leq (3H(a))^n H(Q) \leq (3c_A)^n X^2 \leq X^{2+2\epsilon}. \quad (44)$$

We claim that we have $\varphi(a, \xi) > \kappa + \epsilon$ for at least one choice of $a \in A$ and $\xi \in E$. If we admit this result, then for such choice of a and ξ the polynomial $S(T) = Q(aT)$ and the point ξ have all the required properties. First of all, Lemma 4.2 shows that S is, like Q , a primary polynomial of $\mathbb{Q}[T]$. Its degree is at most n and by (44) its height at most $X^{2+2\epsilon}$. Finally, by definition of $\varphi(a, \xi)$, we also have

$$\frac{|S(\xi)|}{\|S\|} = \frac{|Q(a\xi)|}{\|Q(aT)\|} \leq X^{\epsilon - \varphi(a, \xi)n} \leq X^{-\kappa n}.$$

To prove our claim, we proceed by contradiction assuming on the contrary that φ takes values in $[0, \kappa + \epsilon]$. Let a_1 and a_2 be two distinct elements of A . We apply Proposition 3.1 to the polynomials $Q(a_1T)$ and $Q(a_2T)$ with $s = |E|$ and $t = 1$. Since Q is primary and not a power of T , and since $a_1/a_2 \in \mathbb{Q}_+ \setminus \{1\}$, Lemma 4.2 shows that these polynomials are relatively prime in $\mathbb{Q}[T]$. Therefore, the proposition gives

$$1 \leq cH(Q(a_1T))^n H(Q(a_2T))^n \prod_{\xi \in E} \max \left\{ \frac{|Q(a_1\xi)|}{\|Q(a_1T)\|}, \frac{|Q(a_2\xi)|}{\|Q(a_2T)\|} \right\} \quad (45)$$

where $c = e^{7n^2} (c_E + 2)^{4|E|n} \Delta_E^{-1}$. Using $|E| \leq n$ and the hypotheses (39), we find $c \leq X^{12\epsilon n}$. Substituting this into (45) and using (44) and (42), we find

$$1 \leq X^{12\epsilon n} (X^{2+2\epsilon})^{2n} \prod_{\xi \in E} X^{\epsilon - n \min\{\varphi(a_1, \xi), \varphi(a_2, \xi)\}},$$

and therefore, since $|E| \leq n$ and $\epsilon \leq 1/10$, we finally obtain

$$\sum_{\xi \in E} \min\{\varphi(a_1, \xi), \varphi(a_2, \xi)\} \leq 6 - 3\epsilon.$$

According to Proposition 9.1, this implies that

$$\sum_{a \in A} \sum_{\xi \in E} \varphi(a, \xi) \leq (6 - 3\epsilon) \binom{|A|}{2} + (\kappa + \epsilon)|E| \leq (6 + \kappa - 2\epsilon)|E| < 2\kappa|E|,$$

in contradiction with (43). Therefore φ must take at least one value greater than $\kappa + \epsilon$. $\square$

Proof of Theorem 1.3. It suffices to prove the result in the case where $\alpha = 1$. We proceed by contradiction, assuming on the contrary that for each sufficiently large n there exists a non-zero polynomial $P_n \in \mathbb{Z}[T]$ of degree at most n and height at most $\exp(n^\beta)$ satisfying $\prod_{a \in A_n} \prod_{b \in B_n} |P_n(ab\xi)| \leq \exp(-n^{1+\beta+2\mu+\delta})$. We claim that, for n large enough, all the hypotheses of Proposition 10.1 are satisfied with the choice of $A = A_n$, $E = B_n\xi$, $\epsilon = 1/10$, $X = \exp(n^\beta)$, $P = P_n$ and $\kappa = n^\delta$. First of all the conditions (38) and (40) are fulfilled because the prime number theorem shows that the cardinalities of A_n and B_n behave, respectively, like $n^\mu/(\mu \log n)$ and $n^{2\mu}/(2\mu \log n)$, and we have $2\mu < 1 < \beta$. Finally the condition (39) is also satisfied as we have $c_A \leq n$, $c_E \leq n$ and $\Delta_E \geq 1$ (for n large enough). Therefore, there exist a point $b \in B_n$ and a non-zero primary polynomial S of $\mathbb{Q}[T]$ with

$$\deg(S) \leq n, \quad H(S) \leq \exp(3n^\beta) \quad \text{and} \quad \frac{|S(b\xi)|}{\|S\|} \leq \exp(-n^{1+\beta+\delta}).$$

Upon dividing S by its content, we may assume that $S \in \mathbb{Z}[T]$. Then, assuming again that n is sufficiently large, we deduce that the polynomial $Q = S(bT) \in \mathbb{Z}[T]$ satisfies

$$\deg(Q) \leq n, \quad H(Q) \leq \exp(n^{\beta+\delta/2}) \quad \text{and} \quad |Q(\xi)| \leq \exp(-n^{1+\beta+\delta}),$$

against Gel'fond's Lemma 2.2. $\square$

Proof of Theorem 1.1(3). Suppose on the contrary that, for each sufficiently large n , the polynomial P_n satisfies $|P_n^{[j]}(i\xi)| \leq \exp(-n^\nu)$ for each choice of integers i and j with $1 \leq i \leq n^\sigma$ and $0 \leq j \leq n^\tau$. By [7, Proposition 1], we must have $\sigma > 0$. Define

$$\ell = [4/\sigma] \quad \text{and} \quad \delta = (1/5)(\nu - 1 - \beta + (3/4)\sigma + \tau).$$

For a given integer $n \geq 1$, let $A = A_n$ be the set of all prime numbers p with $p \leq n^{\sigma/4}$, and let $E = AB\xi$ where $B = B_n$ is the set of all prime numbers p with $n^{\sigma/4} < p \leq n^{\sigma/2}$. We claim that, if n is sufficiently large, all the hypotheses of Proposition 8.1 are satisfied with the additional choice of

$$t = [(1 + n^\tau)/2], \quad X = \exp(n^\beta), \quad P = P_n \quad \text{and} \quad \kappa = n^{4\delta}.$$

The conditions (30) are fulfilled because we have $(3/4)\sigma + \tau < 1$ and for large enough values of n the prime number theorem gives

$$n^{\sigma/4}(\log n)^{-1} \leq |A| \leq n^{\sigma/4} \quad \text{and} \quad n^{3\sigma/4}(\log n)^{-2} \leq |E| \leq n^{3\sigma/4}.$$

The conditions (31) are also satisfied because we have $\beta > 1$ and for large enough values of n we find $c_A \leq n$, $c_E \leq n$ and $\delta_E^{-|E|^2 t^2/n} \leq \min(1, |\xi|)^{-n}$ (since $|E|t \leq n$). Finally, as the product AE is contained in $\{\xi, 2\xi, \dots, [n^\sigma]\xi\}$, the hypothesis on P gives

$$\max\{|P^{[j]}(ax)|; a \in A, x \in E, 0 \leq j < 2t\} \leq \exp(-n^\nu) \leq X^{-\kappa n/(t|E|)},$$

and so the main condition (32) is satisfied. Consequently, for each sufficiently large value of n , there exists a non-zero polynomial $S \in \mathbb{Z}[T]$ with

$$\deg(S) \leq \frac{2n}{t|A|} \leq n^{1-\sigma/4-\tau+\delta},$$

$$H(S) \leq \exp\left(\frac{4n^\beta}{t|A|}\right) \leq \exp(n^{\beta-\sigma/4-\tau+\delta}),$$

$$\prod_{a \in A} \prod_{b \in B} |S(ab\xi)| \leq \exp(-n^{1+\beta-2\tau+3\delta}),$$

upon noting, for the last inequality, that any element of E can be written uniquely as a product $ab\xi$ with $a \in A$ and $b \in B$. This contradicts Theorem 1.3 (with $\mu = \sigma/4$). $\square$

11. Higher Transcendence Degree

In this section, we prove the statement (6) of Theorem 1.1 by combining the statement (3) established in the preceding section with the following result.

Proposition 11.1. *Let $n, s \in \mathbb{N}^*$ with $s \leq 2n$, let E and F be finite subsets of $\mathbb{C}$ with $0 \in E$ and $|F| = s$, and let P be any non-zero polynomial of $\mathbb{Z}[T]$ of degree at*

most n . Put

$$\delta_P = \max\{|P(\xi + \eta)|; \xi \in E, \eta \in F\}.$$

Then there exists a non-zero polynomial $R \in \mathbb{Z}[T]$ satisfying

- (i) $\deg(R) \leq n^2$,
- (ii) $H(R) \leq 6^{n^2} H(P)^{2n}$,
- (iii) $\max\{|R^{[k]}(\xi)|; \xi \in E, 0 \leq k \leq [s/2]\} \leq cH(P)^{2n} \min(1, \delta_P)^{s/2}$,

where $c = \Delta_F^{-1} (8^n (1 + c_E)^n (1 + c_F)^s)^{3n}$, $c_E = \max_{\xi \in E} |\xi|$ and $c_F = \max_{\eta \in F} |\eta|$.

Proof. We may assume without loss of generality that P is primitive. Suppose first that its degree is n . We claim that the resultant $R(U)$ of $P(T)$ and $P(T + U)$ with respect to T has the required properties as a polynomial in the new variable U . Since $P(T)$ and $P(T + U)$ are relatively prime elements of $\mathbb{Z}[T, U]$, we know that $R(U)$ is a non-zero polynomial of $\mathbb{Z}[U]$. To prove the estimates (i)–(iii), we apply Lemma 3.4 with $L = \mathbb{Q}(U)$, $m = 2n$, $t = 1$, $Q = 1$, the role of $\xi_1, \dots, \xi_s$ being played by the points $\eta_1, \dots, \eta_s$ of F , and the sequence of polynomials $P_1, \dots, P_m$ being given by

$$P(T), TP(T), \dots, T^{n-1}P(T), P(T + U), TP(T + U), \dots, T^{n-1}P(T + U). \quad (46)$$

In the notation of Lemma 3.4, this gives

$$R(U) = \det(\psi(P_1), \dots, \psi(P_m)) = \pm \Delta^{-1} \det(\varphi(P_1), \dots, \varphi(P_m)) \quad (47)$$

where $\Delta = \prod_{1 \leq i < j \leq s} (\eta_j - \eta_i)$.

To perform the required estimations, we use the following additional notation. For each polynomial G in $\mathbb{C}[U]$ or $\mathbb{C}[T, U]$, we denote by $\|G\|_1$ the sum of the absolute values of its coefficients (its length). For a row vector $G = (G_1, \dots, G_m) \in \mathbb{C}[U]^m$, we denote by $\deg(G)$ the maximum of the degrees of $G_1, \dots, G_m$ and we put $\|G\|_1 = \|G_1\|_1 + \dots + \|G_m\|_1$. We also define $G^{[k]} = (G_1^{[k]}, \dots, G_m^{[k]})$ for each integer $k \geq 0$. We use the same notation for column vectors.

By definition, ψ maps a polynomial $G \in \mathbb{C}[T, U]$ with $\deg_T(G) < m$ to the row vector $\psi(G) \in \mathbb{C}[U]^m$ formed by its coefficients as a polynomial in T over the ring $\mathbb{C}[U]$. Thus we have $\deg(\psi(G)) = \deg_U(G)$ and $\|\psi(G)\|_1 = \|G\|_1$. Applying this to the representation of $R(U)$ given by (47) in terms of ψ , we obtain

$$\begin{aligned} \deg(R) &\leq \sum_{i=1}^m \deg \psi(P_i) = n \deg_U(P(T + U)) = n^2, \\ \|R\|_1 &\leq \prod_{i=1}^m \|\psi(P_i)\|_1 = \|P(T)\|_1^n \|P(T + U)\|_1^n \leq 6^{n^2} H(P)^{2n}, \end{aligned}$$

where the last step uses the crude estimates $\|P(T)\|_1 \leq \|P\| \|(1 + T)^n\|_1 = 2^n H(P)$ and similarly $\|P(T + U)\|_1 \leq \|P\| \|(1 + T + U)^n\|_1 = 3^n H(P)$. This proves (i) and (ii).

For $j = 1, \dots, m$, let $C_j(U)$ denote the j th column of the $m \times m$ matrix with rows $\varphi(P_1), \dots, \varphi(P_m)$. By virtue of (47), we have $R(U) = \pm \Delta^{-1} \det(C_1(U), \dots, C_m(U))$. Using the multi-linearity of the resultant, we deduce that for each integer $k \geq 0$ we have

$$R^{[k]}(U) = \pm \Delta^{-1} \sum_{k_1 + \dots + k_m = k} \det(C_1^{[k_1]}(U), \dots, C_m^{[k_m]}(U)) \quad (48)$$

where the sum runs through all partitions of k into a sum of m non-negative integers $k_1, \dots, k_m$.

For $j = 1, \dots, s$, the transpose of $C_j(U)$ is the row vector formed by the values at η_j of the sequence of polynomials (46):

$${}^t C_j(U) = (P(\eta_j), \dots, \eta_j^{n-1} P(\eta_j), P(U + \eta_j), \dots, \eta_j^{n-1} P(U + \eta_j)).$$

For $\xi \in E$, this gives

$$\|C_j(\xi)\|_1 \leq 2(1 + |\eta_j|)^n \max(|P(\eta_j)|, |P(\xi + \eta_j)|) \leq 2(1 + c_F)^n \delta_P, \quad (49)$$

since both η_j and $\xi + \eta_j$ belong to $E + F$ (as $0 \in E$). For each integer $k \geq 1$, we also find

$${}^t C_j^{[k]}(\xi) = (0, \dots, 0, P^{[k]}(\xi + \eta_j), \dots, \eta_j^{n-1} P^{[k]}(\xi + \eta_j)),$$

and therefore

$$\begin{aligned} \|C_j^{[k]}(\xi)\|_1 &\leq (1 + |\eta_j|)^n |P^{[k]}(\xi + \eta_j)| \\ &\leq (1 + |\eta_j|)^n (1 + |\xi| + |\eta_j|)^n \|P^{[k]}\| \\ &\leq 2^n (1 + c_E)^n (1 + c_F)^{2n} H(P). \end{aligned} \quad (50)$$

For $j = s + 1, \dots, 2n$, the transpose of $C_j(U)$ is the row vector made of the coefficients of T^{j-1} from the polynomials of the sequence (46). It is given by

$${}^t C_j(U) = (P^{[j-1]}(0), \dots, P^{[j-n]}(0), P^{[j-1]}(U), \dots, P^{[j-n]}(U)),$$

with the convention that $P^{[i]} = 0$ when $i < 0$. For each integer $k \geq 1$, this gives

$${}^t C_j^{[k]}(U) = \left(0, \dots, 0, \binom{j+k-1}{j-1} P^{[j+k-1]}(U), \dots, \binom{j+k-n}{j-n} P^{[j+k-n]}(U) \right),$$

with the additional convention that the binomial symbol is zero when its lower entry is negative. From this we deduce that, for each $k \geq 0$ and each $\xi \in E$, we have

$$\|C_j^{[k]}(\xi)\|_1 \leq n 2^n \max_{0 \leq i \leq n} \max(|P^{[i]}(0)|, |P^{[i]}(\xi)|) \leq 2^{3n} (1 + c_E)^n H(P). \quad (51)$$

For each integer k with $0 \leq k \leq s/2$ and each partition of k as a sum of non-negative integers $k_1, \dots, k_m$, there are always at least $s/2$ indices i with $1 \leq i \leq s$ for which $k_i = 0$. Thus, for such k and any $\xi \in E$, the formula (48) combined with (49)–(51) gives

$$\begin{aligned} |R^{[k]}(\xi)| &\leq \Delta_F^{-1} \binom{k+m-1}{m-1} \max_{k_1+\dots+k_m=k} \prod_{j=1}^m \|C_j^{[k_j]}(\xi)\|_1 \\ &\leq \Delta_F^{-1} 2^{3n} (2^{3n} (1+c_E)^n)^{2n} (1+c_F)^{2ns} H(P)^{2n} \min(1, \delta_P)^{s/2}. \end{aligned}$$

This proves (iii) with c replaced by $c' = \Delta_F^{-1} 2^{9n^2} (1+c_E)^{2n^2} (1+c_F)^{2ns}$.

In the general case where P has degree $d \leq n$, we apply the preceding estimates to $\tilde{P}(T) = T^{n-d}P(T)$. Since $\tilde{P}$ has degree n , same height as P , and since it satisfies

$$|\tilde{P}(\xi + \eta)| \leq \max(1, |\xi| + |\eta|)^n \delta_P \leq (1+c_E)^n (1+c_F)^n \delta_P$$

for any $\xi \in E$ and $\eta \in F$, we conclude that the corresponding polynomial R satisfies (i)–(iii) with the given value of c . $\square$

Proof of Theorem 1.1(6). Suppose on the contrary that, for each sufficiently large n , the polynomial P_n satisfies $|P_n(i\xi + \eta)| \leq \exp(-n^\nu)$ for $i = 0, 1, \dots, [n^\sigma]$. If $\sigma = 0$, it follows from Lemma 2.2 that both η and $\xi + \eta$ are algebraic over $\mathbb{Q}$. This is impossible since ξ is transcendental over $\mathbb{Q}$. Thus, we have $\sigma > 0$ and so there exists $\delta > 0$ such that $\sigma > \delta$ and $\nu > 3 + \beta - (11/4)\sigma + 5\delta$. We apply Proposition 11.1 with n replaced by $[\sqrt{n}]$,

$$P = P_{[\sqrt{n}]}, \quad E = \{i\xi; 0 \leq i \leq 2n^{(\sigma-\delta)/2}\} \quad \text{and} \quad F = E + \eta.$$

For n sufficiently large, we have $\max(c_E, c_F) \leq n^{\sigma/2}$, $\Delta_F \geq 1$, $2n^{(\sigma-\delta)/2} \leq |E| = |F| \leq [\sqrt{n}]$, and $\max\{|P(x+y)|; x \in E, y \in F\} \leq \exp(-(1/2)n^{\nu/2})$. So, there exists a non-zero polynomial $R \in \mathbb{Z}[T]$ with $\deg(R) \leq n$, $H(R) \leq \exp(n^{(1+\beta+\delta)/2})$ and

$$\max\{|R^{[j]}(i\xi)|; 0 \leq i, j \leq n^{(\sigma-\delta)/2}\} \leq \exp(-n^{(\nu+\sigma-2\delta)/2}).$$

This contradicts Theorem 1.1(3). $\square$

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Appendix A. Construction of Polynomials with Given Properties

The following result derives from a simple application of Dirichlet box principle.

Proposition A.1. *Let $m \in \mathbb{N}^*$, let $\xi_1, \dots, \xi_m \in \mathbb{C}$, and let $\beta, \sigma_1, \dots, \sigma_m, \tau, \nu > 0$ with $\sigma_1 + \dots + \sigma_m + \tau < 1$ and $1 < \nu < 1 + \beta - \sigma_1 - \dots - \sigma_m - \tau$. For each sufficiently large integer $n \geq 1$, there exists a non-zero polynomial $P_n \in \mathbb{Z}[T]$ of degree at most n and height at most $\exp(n^\beta)$ satisfying $|P_n^{[j]}(i_1\xi_1 + \dots + i_m\xi_m)| \leq \exp(-n^\nu)$ for any choice of integers $i_1, \dots, i_m$ and j with $0 \leq i_1, \dots, i_m \leq n^{\sigma_m}$ and $0 \leq j \leq n^\tau$.*

Proof. For each sufficiently large integer n , the conditions imposed on the polynomial P_n constitute a system of at most $2n^{\sigma_1+\dots+\sigma_m+\tau}$ linear inequations in its $n+1$ unknown coefficients, each having itself complex coefficients of absolute value at most $2^n(n^{\sigma_1}|\xi_1|+\dots+n^{\sigma_m}|\xi_m|)^n \leq \exp(n^\nu)$. The conclusion follows by applying a generic version of Thue–Siegel lemma like [14, Lemma 4.12]. $\square$

In the case where $\sigma_1 = \dots = \sigma_m = \sigma$, the main condition on the parameter ν in Proposition A.1 becomes $\nu < 1 + \beta - m\sigma - \tau$. The next proposition shows that in some instances, for $m \geq 3$, the weaker condition $\nu < 1 + \beta - 2\sigma - \tau$ suffices even for a set of $\mathbb{Q}$ -linearly independent points $\xi_1, \dots, \xi_m$.

Proposition A.2. *Let $m \in \mathbb{N}^*$ with $m \geq 3$ and let $\beta, \sigma, \tau, \nu > 0$ with $2\sigma + \tau < 1$ and $1 < \nu < 1 + \beta - 2\sigma - \tau$. There exist $\mathbb{Q}$ -linearly independent complex numbers $\xi_1, \dots, \xi_m$ with $\xi_1 = 1$, which satisfy the following property. For each sufficiently large integer $n \geq 1$, there exists a non-zero polynomial $P_n \in \mathbb{Z}[T]$ of degree at most n and height at most $\exp(n^\beta)$ satisfying $|P_n^{[j]}(i_1\xi_1 + \dots + i_m\xi_m)| \leq \exp(-n^\nu)$ for any choice of integers $i_1, \dots, i_m$ and j with $0 \leq i_1, \dots, i_m \leq n^\sigma$ and $0 \leq j \leq n^\tau$.*

As the proof will show, these examples are ruled out if we assume that $\xi_1, \dots, \xi_m$ satisfy an appropriate measure of linear independence over $\mathbb{Z}$.

Proof. Without loss of generality, we may assume that $\nu > \beta$. Choose $\delta > 0$ such that $\delta < \sigma$, $m\delta + 2\sigma + \tau < 1$ and $\nu + \delta < 1 + \beta - m\delta - 2\sigma - \tau$. A simple adaptation of the argument of Philippon in the Appendix of [9] (based on a result of Khintchine [5]) provides $\mathbb{Q}$ -linearly independent complex numbers $\xi_1 = 1, \xi_2, \dots, \xi_m$ with the property that, for each integer $n \geq 1$ and each $k = 3, \dots, m$, there exist integers $a_{k,n}$, $b_{k,n}$ and $c_{k,n}$ with

$$\max(|a_{k,n}|, |b_{k,n}|) < |c_{k,n}| \leq n^\delta \quad \text{and} \quad |a_{k,n} + b_{k,n}\xi_2 + c_{k,n}\xi_k| \leq \exp(-n^{\nu+\delta})$$

(the choice of the function $\exp(-n^{\nu+\delta})$ is adapted to our purpose, but any positive valued function of $n \in \mathbb{N}$ would work as well; the only new requirement is the condition $\max(|a_{k,n}|, |b_{k,n}|) < |c_{k,n}|$ which is easily fulfilled). By Proposition A.1, for each n sufficiently large, there exists a non-zero polynomial $P_n \in \mathbb{Z}[T]$ of degree at most n and height at most $\exp(n^\beta)$ such that $|P_n^{[j]}(i_1\xi_1 + \dots + i_m\xi_m)| \leq \exp(-n^{\nu+\delta})$ for any choice of integers $i_1, \dots, i_m$ and j with $0 \leq i_1, i_2 \leq n^{\sigma+\delta}$, $0 \leq i_3, \dots, i_m \leq n^\delta$ and $0 \leq j \leq n^\tau$. We claim that this sequence of polynomials has the required property. To show this, choose integers $n, i_1, \dots, i_m$ and j with $n \geq 1$, $0 \leq i_1, \dots, i_m \leq n^\sigma$ and $0 \leq j \leq n^\tau$. After division, one can write the point $\xi = i_1\xi_1 + \dots + i_m\xi_m$ as a sum $\xi = \xi' + \eta$ where $\xi' = i'_1\xi_1 + \dots + i'_m\xi_m$ and $\eta = \sum_{k=3}^m q_k(a_{k,n} + b_{k,n}\xi_2 + c_{k,n}\xi_k)$ for integers i'_k and q_k satisfying $|i'_k| \leq mn^\sigma$ for $k = 1, 2$, $|i'_k| \leq n^\delta$ for $k = 3, \dots, m$, and $0 \leq q_k \leq n^\sigma$ for $k = 3, \dots, m$. For n large enough, this gives

$$\begin{aligned} |P_n^{[j]}(\xi)| &\leq |P_n^{[j]}(\xi')| + |\eta| \|P_n^{[j]}\| (1 + |\xi| + |\xi'|)^n \\ &\leq \exp(-n^{\nu+\delta}) + (mn^\sigma \exp(-n^{\nu+\delta})) (2^n \exp(n^\beta)) n^n \\ &\leq \exp(-n^\nu). \end{aligned}$$

 $\square$

Appendix B. A Note on Intersection Estimates

We prove the following result as a complement to the estimates of Sec. 6.

Proposition B.1. *Let s be a positive integer and let F be a non-empty finite subset of $\mathbb{Z}^s$. Define $E = F \cap (F - \mathbf{e}_1) \cap \cdots \cap (F - \mathbf{e}_s)$, where $(\mathbf{e}_1, \dots, \mathbf{e}_s)$ denotes the canonical basis of $\mathbb{Z}^s$. Then we have $|E| \leq |F| - |F|^{(s-1)/s}$.*

Proof. We proceed by induction on s . If $s = 1$, we have $F \neq F - \mathbf{e}_1$ and so we get $|E| \leq |F| - 1$ as stated. Assume from now on that $s \geq 2$ and that the result holds in dimension $s - 1$. For each $i \in \mathbb{Z}$, we define

$$E_i = \{\mathbf{x} \in \mathbb{Z}^{s-1}; (\mathbf{x}, i) \in E\} \quad \text{and} \quad F_i = \{\mathbf{x} \in \mathbb{Z}^{s-1}; (\mathbf{x}, i) \in F\}.$$

Let I denote the set of indices $i \in \mathbb{Z}$ such that $F_i \neq \emptyset$, and write $(\mathbf{e}'_1, \dots, \mathbf{e}'_{s-1})$ for the canonical basis of $\mathbb{Z}^{s-1}$. For $i \in I$ we have $E_i \subseteq F_i \cap (F_i - \mathbf{e}'_1) \cap \cdots \cap (F_i - \mathbf{e}'_{s-1})$ and so the induction hypothesis gives $|E_i| \leq |F_i| - |F_i|^{(s-2)/(s-1)}$. Summing on $i \in I$, upon noting that $E_i = \emptyset$ when $i \notin I$, this gives $|E| \leq |F| - S$ where $S = \sum_{i \in I} |F_i|^{(s-2)/(s-1)}$. We also have $E_i \subseteq F_{i+1}$ for each $i \in \mathbb{Z}$, thus $E_i \subseteq F_i \cap F_{i+1}$ and so

$$|E| \leq \sum_{i \in \mathbb{Z}} |F_i \cap F_{i+1}| = \sum_{i \in \mathbb{Z}} (|F_i| - |F_i \setminus F_{i+1}|) \leq |F| - \left| \bigcup_{i \in \mathbb{Z}} (F_i \setminus F_{i+1}) \right|.$$

Since $\cup_{i \in \mathbb{Z}} (F_i \setminus F_{i+1}) = \cup_{i \in \mathbb{Z}} F_i$ contains each F_i , we deduce that $|E| \leq |F| - M$ where $M = \max_{i \in I} |F_i|$. By definition of S and M , we have $SM^{1/(s-1)} \geq \sum_{i \in I} |F_i| = |F|$, and so we get $|E| \leq |F| - \max(S, M) \leq |F| - |F|^{(s-1)/s}$ as required. $\square$

Corollary B.2. *With the same notation, we have $|E| \leq (1/s)|F| \log |F|$.*

Proof. Define $g(x) = |F|^x$ for each $x > 0$. The proposition gives $|E| \leq g(1) - g((s-1)/s)$, thus $|E| \leq (1/s)g'(\theta) = (1/s)|F|^\theta \log |F|$ for some real number θ in the interval $((s-1)/s, 1)$. Since $|F|^\theta \leq |F|$, the conclusion follows. $\square$

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